# Exercise 1

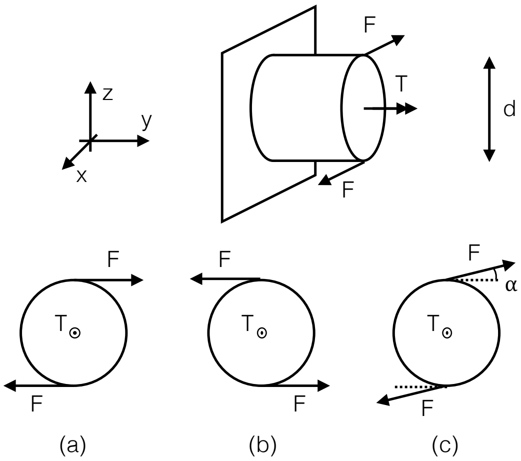


Figure 1: Simple cylindric bar under torsion, three different cases viewed from the right side.

A simple cylindric bar is clamped at one end and put under torsion by a force couple (fig. 1, forces F). As seen in class, these two forces can also be expressed as an applied torque (fig. 1, torque T).

For each of the three cases, express T as a function of F. Respect the sign conventions. What is special about case (c)?

*Reminder: the moment (or torque) of a force is defined according to fig. 2 and is given by:*

$$\vec{T} = \vec{r} \times \vec{F} \rightarrow T = rF_{\perp} = rF\sin\theta$$

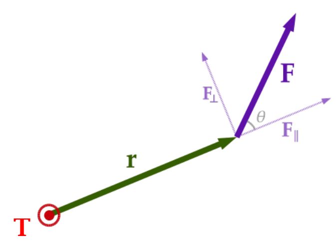


Figure 2: Definition of a torque.

## Exercise solution 1

### Given:

- Diameter of the bar: $d$
- Force magnitude: $F$

### Asked:

- Equivalent torque $T$ for all three cases.

### Relevant relationships:

- Moment of a force:

  $$\overrightarrow{T_1} = \overrightarrow{r} \times \overrightarrow{F}; T_1 = rF\sin\theta$$

- Moment of a force couple:

  $$\overrightarrow{T_2} = 2\overrightarrow{r} \times \overrightarrow{F}; T_2 = 2rF\sin\theta = dF\sin\theta$$

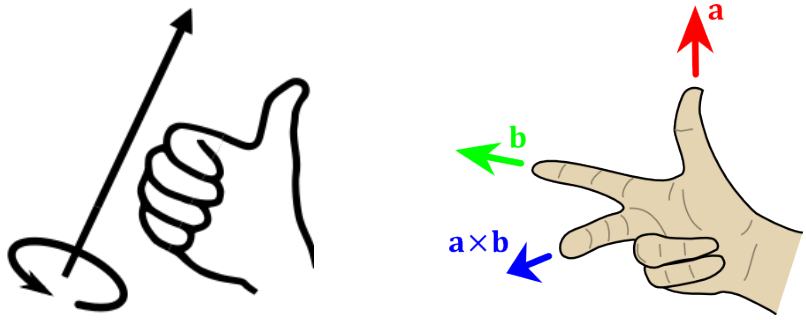


Figure 3: Right hand rule.

The resolution is straightforward: we only need to apply the vector cross product. To find the sign of the torque, we apply the right hand rule (fig. 3). We obtain:

- (a) We apply the right hand rule and find that T must be negative. Since $\overrightarrow{r}$ and $\overrightarrow{F}$ are perpendicular, $\sin\theta = 1$:

  $$T = -2\frac{d}{2}F = -d \cdot F$$

- (b) This time, T is positive:

  $$T = 2\frac{d}{2}F = d \cdot F$$

- (c) In this case, $\overrightarrow{r}$ and $\overrightarrow{F}$ are not perpendicular. This means that the cylinder is subject to torsion and also to an elongation in the z-axis.

  $$T = -2\frac{d}{2}F\sin\theta = -d \cdot F\cos\alpha$$

## Exercise 2

A patient has had to amputate his leg and will receive a prosthesis. He used to play basketball and would like to continue to do so. The prosthesis should be built in order to support his movements. During one move, when he wants to pivot on one leg, the forces depicted in figure 4 will act on the prosthetic foot. To simplify the scenario, assume that the forces have their lines of action at a distance $b = 100$mm from the outside of the tube, as shown in figure 4.

The allowable shear stress in the tube is 400MPa. The inner radius of the prosthetic tube is $r_1 = 10$mm and the outer radius is $r_2 = 15$mm.

Calculate the maximum force that the patient can apply as depicted in figure 4, before the prosthesis breaks.

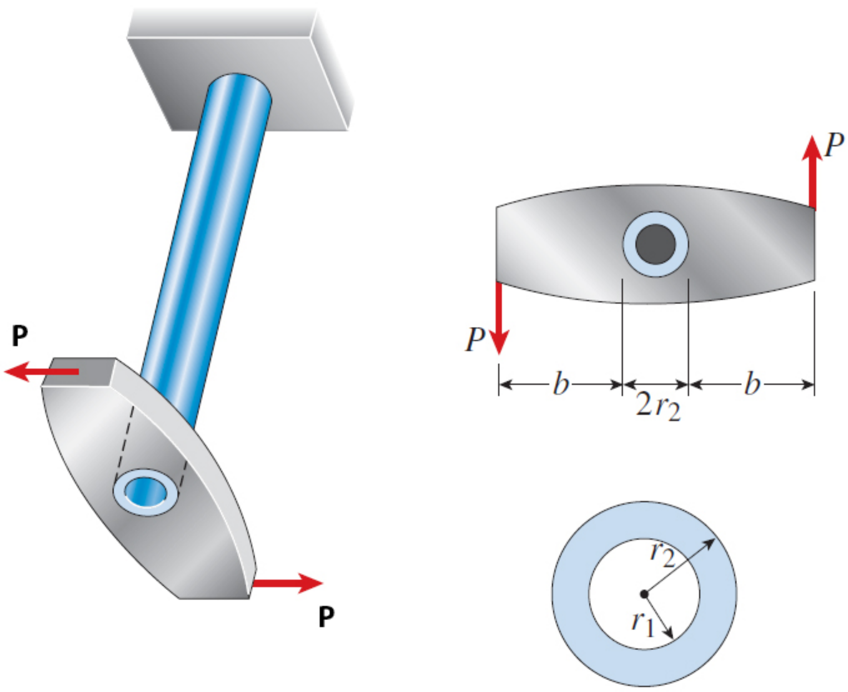


Figure 4: A very simplified foot prosthesis.

## Exercise solution 2

**Given:**

- Distance from center, where the forces are acting

- Inner and outer radius of the prosthesis tube
- Allowable shear stress $\tau_{\max}$

**Asked:**

- Maximum applicable force $P$

**Relevant relationships:**

- Torsion formula $T = P \cdot c$; with c = distance from center
- Second moment of inertia $I_p$

**Solution:** First we have to calculate the torque that is generated by the forces. Since the forces are acting at the distance $b + r_2$ from the center, we have

$$T = 2P(b + r_2) \tag{1}$$

The second moment of inertia for a tube is given by the equation 2, as seen in the course.

$$I_{p_{\text{tube}}} = \frac{\pi}{2}\left(r_2^4 - r_1^4\right) \tag{2}$$

The maximum shear stress is given by

$$\tau_{\max} = \frac{Tr_2}{I_{p_{\text{tube}}}} = \frac{2P(b + r_2)r_2}{\frac{\pi}{2}\left(r_2^4 - r_1^4\right)} = \frac{4P(b + r_2)r_2}{\pi\left(r_2^4 - r_1^4\right)} \tag{3}$$

By solving after $P$ we have

$$\begin{aligned} 4P(b + r_2)r_2 &= \tau_{\max} \cdot \pi\left(r_2^4 - r_1^4\right) && (4) \\ P &= \frac{\tau_{\max} \cdot \pi\left(r_2^4 - r_1^4\right)}{4(b + r_2)r_2} && (5) \\ &= \frac{400 \cdot 10^6\text{N/m}^2 \cdot \pi\left(\left(15 \cdot 10^{-3}\text{m}\right)^4 - \left(10 \cdot 10^{-3}\text{m}\right)^4\right)}{4((100 \cdot 10^{-3}\text{m} + 15 \cdot 10^{-3}\text{m}) \cdot (15 \cdot 10^{-3}\text{m})} && (6) \\ &= 7398.678\text{N} && (7) \end{aligned}$$

Written with significant digits, the patient can apply a maximum force of

$$\boxed{P = 7.4\text{kN}}$$

## Exercise 3

A tapered bar $AB$ of solid circular cross section is twisted by torques $T$ (see fig. 5). The diameter of the bar varies linearly from $d_A$ at the left-hand end to $d_B$ at the right-hand end. The length of the bar is $L$ and the shear modulus of the bar material is $G$.

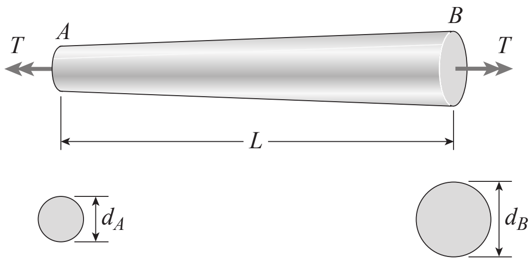


Figure 5: A tapered bar of solid circular cross section.

a) Derive the formula for the angle of twist of the tapered bar $AB$.

b) We know that a slightly different solid circular bar of diameter $d_A$ and length $L$, made of the same material and under the same torques $T$, has an angle of twist $\varphi = 0.1\,\text{rad}$. Knowing also that $\frac{d_B}{d_A} = \beta = 1.5$, calculate the angle of twist of the tapered bar.

## Exercise solution 3

**Given:**

- Diameter at the left-hand end: $d_A$
- Diameter at the right-hand end: $d_B$
- Bar length: $L$
- Shear modulus of the bar material: $G$
- Applied torque: $T$
- Diameter varies linearly along the length of the bar
- Angle of twist of the solid circular bar, having diameter $d_A$ and length $L$: $\varphi = 0.1\,\text{rad}$
- Ratio: $\frac{d_B}{d_A} = \beta = 1.5$

**Asked:**

- Formula for the angle of twist $\varphi_A$ of the tapered bar
- The value of the angle of twist $\varphi_A$ of the tapered bar

**Relevant relationships:**

- Second moment of inertia for circle cross section:

$$J = \frac{\pi R^4}{2}$$

- Angle of twist:

$$\varphi = \int \frac{T}{GJ(x)} \,\mathrm{d}x$$

**Solution:**

## a)

The radius of the tapered bar is:

$$r(x) = \frac{(d_B - d_A)}{2L} x + \frac{d_A}{2}$$

Then the second moment of inertia of the tapered bar is:

$$J(x) = \frac{\pi (r(x))^4}{2}$$

The angle of twist of the tapered bar is :

$$\varphi = \frac{T}{G} \int_0^L \frac{\mathrm{d}x}{J(x)} = \frac{2T}{\pi G} \int_0^L \frac{\mathrm{d}x}{(r(x))^4}$$

We will now perform the substitution in the integral:

$$r = \frac{(d_B - d_A)}{2L} x + \frac{d_A}{2},\ \mathrm{d}r = \frac{(d_B - d_A)}{2L} \,\mathrm{d}x$$

After the substitution we get:

$$\varphi = \frac{2T}{\pi G} \int_{d_A/2}^{d_B/2} \frac{2L}{d_B - d_A} \cdot \frac{\mathrm{d}r}{r^4} = \frac{4TL}{\pi G (d_b - d_A)} \cdot \frac{-1}{3} \cdot \left( \frac{8}{d_B^3} - \frac{8}{d_A^3} \right)$$

Therefore we can write angle of twist as:

$$\varphi = \frac{32TL}{3\pi G} \frac{1}{(d_B - d_A)} \left( \frac{1}{d_A^3} - \frac{1}{d_B^3} \right) = \frac{32TL}{3\pi G} \frac{1}{(d_B - d_A)} \frac{(d_B - d_A)(d_B^2 + d_B d_A + d_A^2)}{d_A^3 d_B^3}$$

$$\varphi = \frac{32TL}{3\pi G} \frac{(d_B^2 + d_B d_A + d_A^2)}{d_A^3 d_B^3}$$

## b)

The second moment of inertia for circle cross section is:

$$J = \frac{\pi d^4}{32}$$

The angle of twist of the solid circular bar is:

$$\varphi_1 = \frac{32TL}{\pi G d_A^4}$$

So we can write write torque $T$ as:

$$T = \frac{\varphi_1 \pi G d_A^4}{32L}$$

If we substitute that in the formula for the angle of twist for the tapered bar $\varphi$ obtained in a) we get:

$$\varphi = \frac{\varphi_1 d_A^4}{3} \frac{(d_B^2 + d_B d_A + d_A^2)}{d_A^3 d_B^3}$$

This we can rewrite as:

$$\varphi = \frac{\varphi_1}{3} \frac{d_A^3}{d_B^3} \frac{(d_B^2 + d_B d_A + d_A^2)}{d_A^2} = \frac{\varphi_1}{3} \frac{1}{\beta^3} (\beta^2 + \beta + 1)$$

From there we get the value of $\varphi$:

$$\varphi = 0.0469\,\mathrm{rad}$$

# Exercise 4

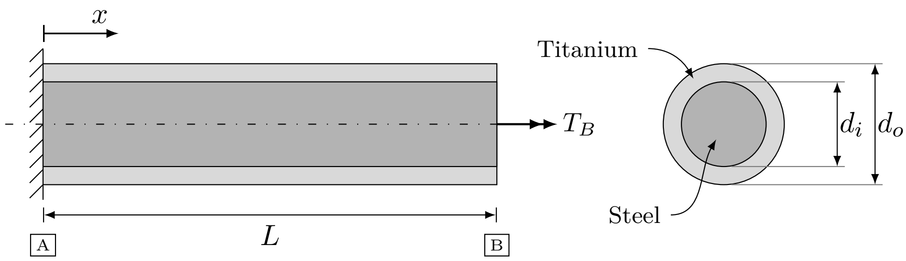


Figure 6: Composite bar made of an steel core with a titanium cladding under torsion.

A steel bar has been made biocompatible by cladding it with a hull of titanium (fig. 6). The part is then subjected to a torque $T_B = 20\,\mathrm{N\,m}$. The steel core has a diameter of 8 mm, the cladding has an outer diameter of 10 mm and the whole bar has a length of 100 mm. Steel has a shear modulus $G_{\mathrm{Steel}}$ of 79 GPa and titanium has a shear modulus $G_{\mathrm{Ti}}$ of 41 GPa.

a) Find the torsional stiffness (torsional spring constant) of the beam, as well as the twist in point B.

b) Calculate the minimal and maximal shear stresses in the steel and the titanium part.

c) Draw the radial shear stress distribution $\tau(r)$ over the whole part.

## Exercise solution 4

**Given:**

- Diameter of the steel bar: $d_i = 8\,\text{mm}$
- Outer diameter of the cladded titanium $d_o = 10\,\text{mm}$
- Bar length: $L = 100\,\text{mm}$
- Shear modulus of steel: $G_{\text{Steel}} = 79\,\text{GPa}$
- Shear modulus of titanium: $G_{\text{Ti}} = 41\,\text{GPa}$
- Applied torque: $T_B = 20\,\text{N m}$

**Asked:**

- Twist in point B.
- Minimal and maximal shear stresses in the steel and the titanium part.
- Sketch of the radial shear stress distribution $\tau(r)$ over the whole part.

**Relevant relationships:**

- Second moment of inertia:
$$J = \int r^2 \,\mathrm{d}A$$
- Angle of twist:
$$\varphi_i = \frac{T_i L_i}{G_i J_i}$$
- Maximum shear stress:
$$\tau_{max} = \frac{T r_{max}}{J}$$

**Solution:**

### a)

The part consists of a steel bar and a titanium tube. The twist at the end of the bar has to be the same for both materials, and the required torque therefore the sum of the two individual torques:

$$T_B = T_{\text{Steel}} + T_{\text{Ti}} = \frac{G_{\text{Steel}} J_{\text{Steel}} \varphi}{L} + \frac{G_{\text{Ti}} J_{\text{Ti}} \varphi}{L} = \frac{(G_{\text{Steel}} J_{\text{Steel}} + G_{\text{Ti}} J_{\text{Ti}})}{L} \varphi$$

For each one we have to find the second moment of inertia.
For the steel core it is straight forward:

$$J_{\text{Steel}} = \frac{\pi d_i^4}{32} = \frac{\pi\,(8\,\text{mm})^4}{32} = 402.1\,\text{mm}^4$$

For the titanium tube we need to find the second moment of inertia of tube with inner diameter of $d_i$ and outer diameter of $d_o$:

$$J = \int_{d_i/2}^{d_o/2} r^2 2\pi r\,\text{d}r = 2\pi \int_{d_i/2}^{d_o/2} r^3\,\text{d}r = \frac{\pi}{32}\left(d_o^4 - d_i^4\right)$$

$$J_{\text{Ti}} = \frac{\pi}{32}\left((10\,\text{mm})^4 - (8\,\text{mm})^4\right) = 579.6\,\text{mm}^4$$

So the torsional spring constant is:

$$k = \frac{(G_{\text{Steel}}J_{\text{Steel}} + G_{\text{Ti}}J_{\text{Ti}})}{L} = 555.3\,\text{N}\,\text{m}\,\text{rad}^{-1}$$

and the finally the twist:

$$\varphi = \frac{T_B L}{G_{\text{Steel}}J_{\text{Steel}} + G_{\text{Ti}}J_{\text{Ti}}} = \frac{20\,\text{N}\,\text{m}}{554.6\,\text{N}\,\text{m}\,\text{rad}^{-1}} = 0.0360\,\text{rad}$$

which is equivalent to 2.07°.

## b)

The torque in the structures is different, so we need the individual contributions:

$$T_{\text{Ti}} = \frac{(G_{\text{Ti}}J_{\text{Ti}})}{L}\cdot\varphi = \frac{(G_{\text{Ti}}J_{\text{Ti}})}{G_{\text{Steel}}J_{\text{Steel}} + G_{\text{Ti}}J_{\text{Ti}}}\cdot T_B = 0.428T_B = 8.56\,\text{N}\,\text{m}$$

and directly

$$T_{\text{Steel}} = T_B - T_{\text{Ti}} = 11.44\,\text{N}\,\text{m}$$

The minimum shear stress in the steel tube is zero along its axis and takes its maximum at $\frac{d_i}{2}$:

$$\tau_{max} = \frac{T_{\text{Steel}}d_i}{2J_{\text{Steel}}} = \frac{11.43\,\text{N}\,\text{m}\times 8\,\text{mm}}{2\times 401.2\,\text{mm}^4} = 114\,\text{MPa}$$

The minimum and maximum shear stress in the titanium tube:

$$\tau_{min} = \frac{T_{\text{Ti}}d_i}{2J_{\text{Ti}}} = \frac{8.56\,\text{N}\,\text{m}\times 8\,\text{mm}}{2\times 579.6\,\text{mm}^4} = 59.1\,\text{MPa}$$

$$\tau_{max} = \frac{T_{\text{Ti}}d_o}{2J_{\text{Ti}}} = \frac{8.56\,\text{N}\,\text{m}\times 10\,\text{mm}}{2\times 579.6\,\text{mm}^4} = 73.9\,\text{MPa}$$

## c)

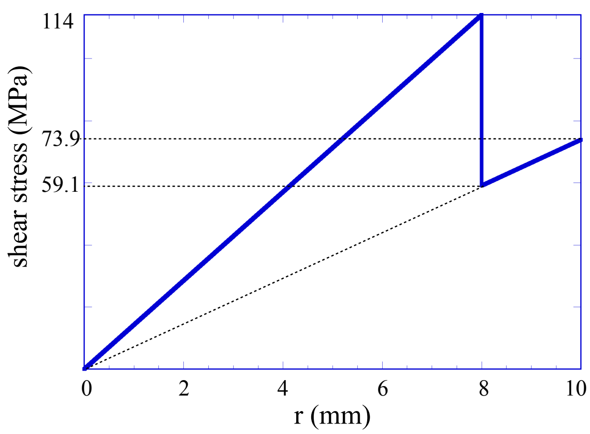


Figure 7: Sketch of the radial shear stress distribution $\tau(r)$ .