Exercise 1

In a titanium structure you have identified the critical point where you know that your part will start to fail. You have calculated the local stress state to be

$$\overleftrightarrow{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \begin{bmatrix} 85 & 0 & -15\sqrt{3} \\ 0 & -20 & 0 \\ -15\sqrt{3} & 0 & 55 \end{bmatrix} \text{MPa}$$

You know that the titanium you are using has a yield strength of 180 MPa. Use the von Mises stress criterion to determine if your part will fail, and if not how big your safety factor is.

Exercise solution 1

Given:

- Stress tensor in critical point.
- Yield stress in material.

Asked:

- Von Mises equivalent stress σ_M.
- Does structure yield? Safety factor.

Relevant relationships:

- Von Mises equivalent stress

  $$\sigma_M = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]}$$

  for principal stresses $\sigma_{1,2,3}$

By looking at the given stress tensor, we can see that there are no shears acting in y direction from either x or z, thus we already know that $\sigma_2 = -20\,\text{MPa}$. This allows us to further treat the remaining part of the tensor

$$\overleftrightarrow{\tau_r} = \begin{bmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{bmatrix} = \begin{bmatrix} 85 & -15\sqrt{3} \\ -15\sqrt{3} & 55 \end{bmatrix} \text{MPa}$$

Finding the eigenvalue solves to

$$\det\left(\overleftrightarrow{\tau_r} - \lambda \overleftrightarrow{E}\right) = 0$$

$$\lambda_1 = \sigma_1 = 40\,\text{MPa} \text{ and } \lambda_2 = \sigma_3 = 100\,\text{MPa}$$

Using these principal stresses give σ_M as:

$$\sigma_M = \sqrt{\frac{1}{2}\left[(40+20)^2 + (-20-100)^2 + (40-100)^2\right]}\text{MPa}$$
$$= 60\sqrt{3}\text{ MPa} = 103.9\,\text{MPa}$$

So we know the structure does not fail and we get a safety factor of

$$\text{SF} = \frac{180}{104} \approx 1.7$$

Exercise 2

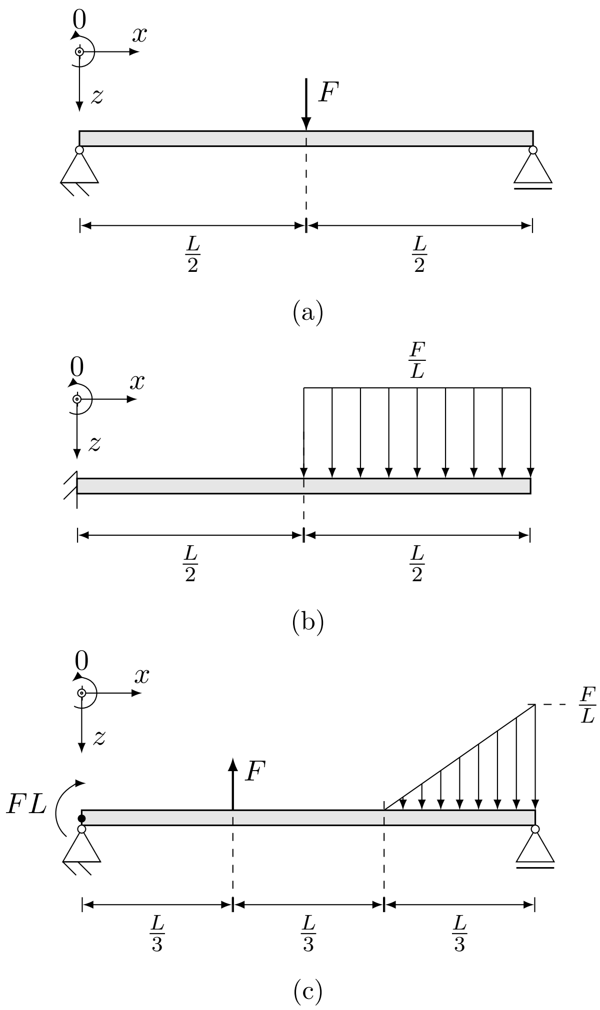


Figure 1: Different loaded beams.

The three beams as shown in figure 1 are subjected to different loads and supports. For each beam shown:

- Find the distributed load function $q(x)$.
- Determine the boundary conditions for the shear force and the moment function.
- Calculate and sketch the shear force $V(x)$ and the internal moment $M(x)$.

Exercise solution 2

Given: Beams with loads.

Asked:

- Distributed load $q(x)$.
- Boundary conditions.
- Shear force and moment functions and diagrams.

Relevant relationships:

- Shear force
$$\frac{\mathrm{d}V(x)}{\mathrm{d}x} = -q(x)$$
- Bending moment
$$\frac{\mathrm{d}M(x)}{\mathrm{d}x} = V(x)$$

a)

The only load that should be used in the load function is the force acting in the middle of the beam, as the supports are on the boundary. The force is modelled with a singularity function to get

$$q_a(x) = F\left\langle x - \tfrac{L}{2}\right\rangle^{-1}.$$

Since there are two pin supports on either end of the beam, the shear forces are as of yet unknown. However, neither end can support a bending moment, thus we get $M_a(0) = 0$ and $M_a(L) = 0$.
Shear force and moment functions are calculated by integration:

$$V_a(x) = -\int q_a(x)\,\mathrm{d}x + C_1 = -F\left\langle x - \tfrac{L}{2}\right\rangle^{0} + C_1$$
$$M_a(x) = \int V(x)\,\mathrm{d}x + C_2 = -F\left\langle x - \tfrac{L}{2}\right\rangle^{1} + C_1 x + C_2$$

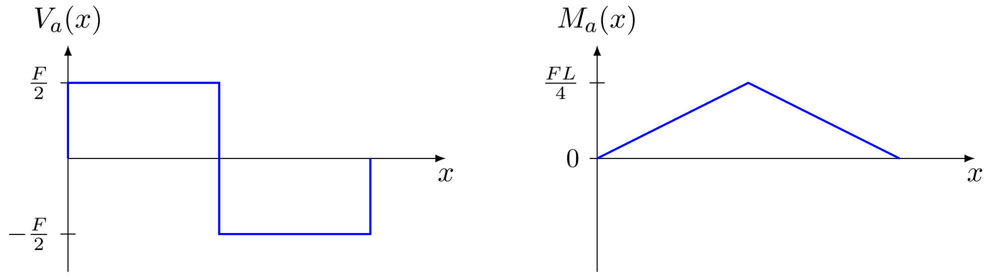
for which we use the boundary conditions to determine C_1 and C_2. For $M_a(0) = 0$ we directly get $C_2 = 0$ and with $M_a(L) = 0$

$$-F\left(L - \tfrac{L}{2}\right) + C_1 L = 0 \quad \rightarrow \quad C_1 = \frac{F}{2}$$

so we finally get

$$V_a(x) = -F\left\langle x - \tfrac{L}{2}\right\rangle^0 + \frac{F}{2}$$
$$M_a(x) = -F\left\langle x - \tfrac{L}{2}\right\rangle^1 + \frac{F}{2}\cdot x$$

which is sketched below:



Note that the two supports each induce a reaction force of $-\frac{F}{2}$, bringing the shear force diagram to 0!

b)

Again reaction forces and moments (from the wall) are neglected and only the line force is used to get

$$q_b(x) = \frac{F}{L}\left\langle x - \tfrac{L}{2}\right\rangle^0.$$

The shear force and moment induced by the wall is yet unknown, but the free right end can support neither a force nor a moment, so $V_b(L) = 0$ and $M_b(L) = 0$.
Again integrate to get the shear force and bending moment functions:

$$V_b(x) = -\int q_a(x)\,\mathrm{d}x + C_1 = -\frac{F}{L}\left\langle x - \tfrac{L}{2}\right\rangle^1 + C_1$$
$$M_b(x) = \int V(x)\,\mathrm{d}x + C_2 = -\frac{F}{2L}\left\langle x - \tfrac{L}{2}\right\rangle^2 + C_1 x + C_2$$

Using first $V_b(L) = 0$ we get

$$-\frac{F}{L}\left(L - \tfrac{L}{2}\right)^1 + C_1 \quad \rightarrow \quad C_1 = \frac{F}{2}$$

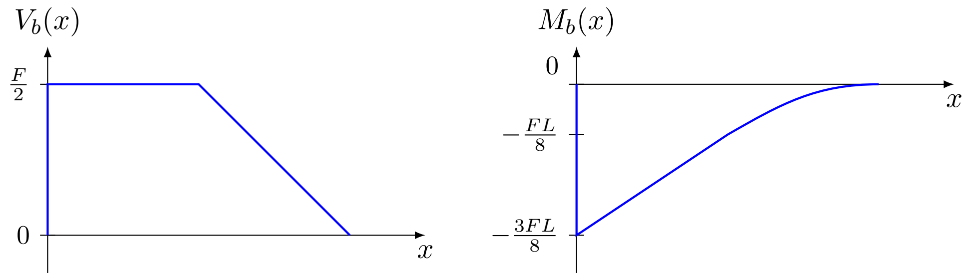
and using $M_b(L) = 0$

$$-\frac{F}{2L}\left(L - \tfrac{L}{2}\right)^2 + \frac{F}{2}\cdot L + C_2 = -\frac{F}{2L}\frac{L^2}{4} + \frac{FL}{2} + C_2 = 0 \quad \rightarrow \quad C_2 = -\frac{3FL}{8}$$

so the complete functions are

$$V_b(x) = -\frac{F}{L}\left\langle x - \tfrac{L}{2}\right\rangle^1 + \frac{F}{2}$$
$$M_b(x) = -\frac{F}{2L}\left\langle x - \tfrac{L}{2}\right\rangle^2 + \frac{F}{2}x - \frac{3FL}{8}$$

which is shown below:



Again the boundary force and moment can be seen from the functions and the graph. The wall reaction force on the beam is $-V_b(0) = -\frac{F}{2}$ in z direction and the reaction moment is $-M_b(0) = \frac{3}{8}FL$.

c)

The load function here is composed of two parts: a point load at $x = \frac{1}{3}L$ and a linearly distributed load that is 0 at $x = \frac{2}{3}L$ and increases linearly to $\frac{F}{L}$ at $x = \frac{1}{3}L$. We write the load function as

$$q_c(x) = -F\left\langle x - \tfrac{L}{3}\right\rangle^{-1} + \frac{3F}{L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^{1}$$

Additionally there is a point moment $M = -FL$ acting on the left wall. This could either be modeled at a point moment while treating the left end as unable to support a moment ($q(x) = M \cdot \langle x\rangle^{-2}$ together with $M_c(0) = 0$) or by directly including it as a boundary condition $M_c(0) = FL$. The right end gives $M_c(L) = 0$ in analogy to a).
Integration yields

$$V_c(x) = -\int q_a(x)\,\mathrm{d}x + C_1 = +F\left\langle x - \tfrac{L}{3}\right\rangle^{0} - \frac{3F}{2L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^{2} + C_1$$
$$M_c(x) = \int V(x)\,\mathrm{d}x + C_2 = F\left\langle x - \tfrac{L}{3}\right\rangle^{1} - \frac{3F}{6L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^{3} + C_1 x + C_2$$

The constants are again determined by the boundary conditions

$$M(0) = FL: \quad C_2 = FL$$

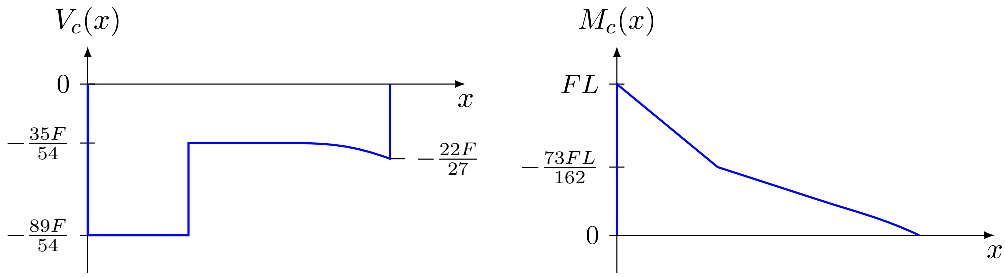
and

$$M(L) = 0: \quad F\left(L - \tfrac{L}{3}\right) - \frac{1F}{2L^2}\left(L - \tfrac{2L}{3}\right)^3 + C_1 L + FL = 0 \quad \rightarrow \quad C_1 = -\frac{89F}{54}$$

With subsituted constants

$$V_c(x) = F\left\langle x - \tfrac{L}{3}\right\rangle^{0} - \frac{3F}{2L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^{2} - \frac{89F}{54}$$
$$M_c(x) = F\left\langle x - \tfrac{L}{3}\right\rangle^{1} - \frac{3F}{6L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^{3} - \frac{89F}{54}\cdot x + FL$$

as drawn below:



Exercise 3

A uniform beam is supported and loaded according to the figure 2.

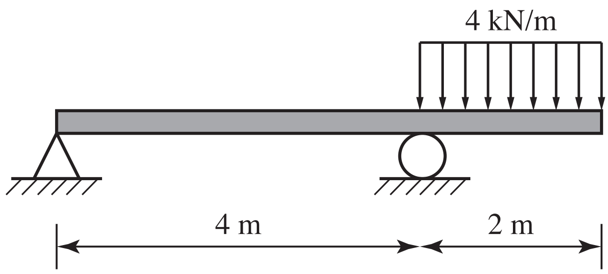


Figure 2: A supported beam under the distributed load

a) Sketch the free body diagram

b) Find reaction forces at the supports

c) Draw the shear force and moment diagram

Exercise solution 3

Given:

- Sketch of the beam
- Distributed load $q_0 = 4\,\mathrm{kN\,m^{-1}}$
- Distance $L_1 = 4\,\mathrm{m}$
- Distance $L_2 = 2\,\mathrm{m}$

Asked:

- Free body diagram
- Reaction forces at the supports
- Shear force diagram
- Moment diagram

Relevant relationships:

- Equilibrium of force

$$\sum F_z = 0 \quad (1)$$

- Equilibrium of moment

$$\sum M_B = 0 \quad (2)$$

- Relation between shear forve and bending moment

$$M(x) = \int V(x)\,\mathrm{d}x + C \quad (3)$$

- Relation between distrinuted load and shear force

$$V(x) = -\int q(x)\,\mathrm{d}x + C \quad (4)$$

a)

Figure 3 presents the free body diagram.

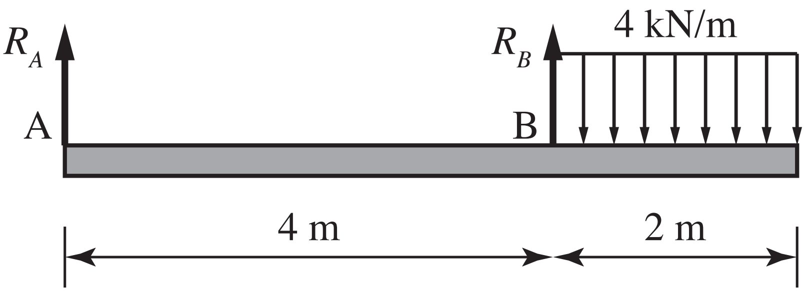


Figure 3: The free body diagram

b)

From force equilibrium: $\sum F_z = 0$ we get

$$-R_A - R_B + q_0 L_2 = 0 \quad (5)$$

$$R_B = q_0 L_2 - R_A \quad (6)$$

From moment equilibrium around point B: $\sum M_B = 0$ we get

$$R_A L_1 + q_0 L_2 \frac{L_2}{2} = 0 \quad (7)$$

$$R_A = -q_0 \frac{L_2^2}{2L_1} = -2\,\mathrm{kN} \quad (8)$$

$$R_B = 10\,\mathrm{kN} \quad (9)$$

c)

We will make two sections: before point B (part 1.) and after point B (part 2). From force equilibria for each section we get:

$$V_1(x) - R_A = 0 \quad (10)$$
$$V_2(x) - R_A - R_B + q_0 (x - L_1) = 0 \quad (11)$$
$$(12)$$

From previous equations we get

$$V_1(x) = R_A = -2\,\text{kN} \quad (13)$$
$$V_2(x) = R_A + R_B - q_0 (x - L_1) = 8\,\text{kN} - 4\,\text{kN}\,\text{m}^{-1} (x - 4\,\text{m}) \quad (14)$$
$$(15)$$

Now we can draw the shear force diagram, as presented in figure 4a.

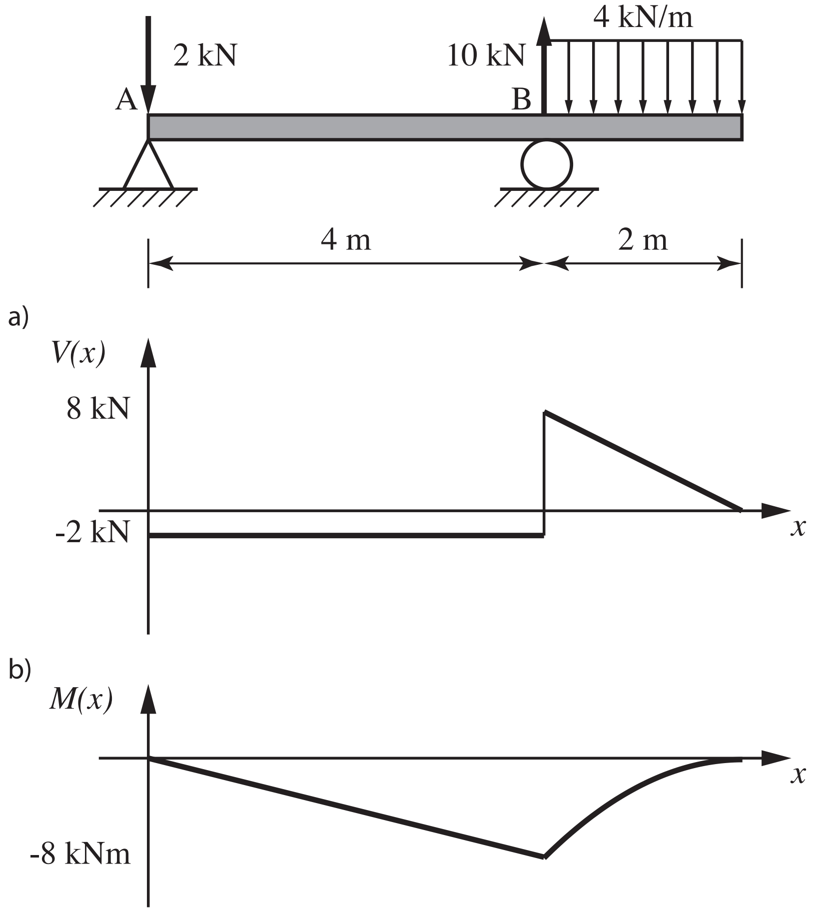


Figure 4: a) Shear force moment diagram B) Moment diagram

The moment diagram we can calcuate by integration of the shear force diagram.

For part 1:

$$M_1(x) = \int R_A \,\mathrm{d}x + C_1 = R_A x + C_1 \tag{16}$$

Knowing that $M_1(0) = 0$ (moment is 0 at pin support) we get $C_1 = 0$ and $M_1(x) = -2\,\mathrm{kN} \cdot x$.

For part 2:

$$M_2(x) = \int \left(R_A + R_B - q_0\,(x - L_1)\right) \mathrm{d}x + C_2 \tag{17}$$

$$M_2(x) = (R_A + R_B)\,x - q_0 \frac{(x - L_1)^2}{2} + C_2 \tag{18}$$

Value of C_2 we can get from condition $M_1(L_1) = M_2(L_1)$, from which we get $C_2 = -40\,\mathrm{kN\,m}$
Finally, we can write

$$M_2(x) = 8\,\mathrm{kN} \cdot x - 2\,\mathrm{kN\,m^{-1}}\,(x - 4\,\mathrm{m})^2 - 40\,\mathrm{kN\,m} \tag{19}$$

From these calculations we can now draw moment diagram, as presented on the figure 4b.