ME – 221
MATLAB® PROBLEM SET 2

You are expected to upload 2 files:

1. A **report** (.pdf file) that addresses the questions by showing the equations derived, the plots generated by the MATLAB script, and your interpretation of the results.

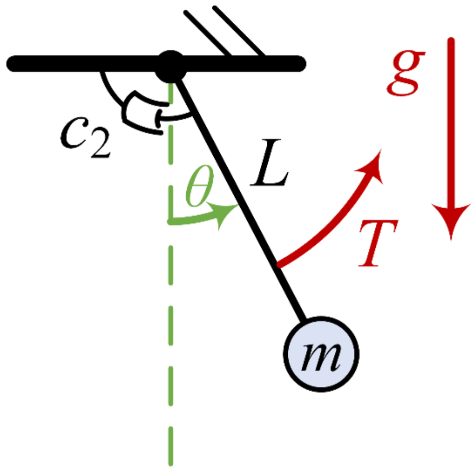
2. The **MATLAB files** (.m files) written to simulate the systems.

File naming: Surname_SCIPER_Assignment2.pdf and Surname_SCIPER_Assignment2.m

Important!

- All plots, developments, and results must appear in the report.
- When using `ode45` you may need to adjust the error tolerance
```
opts = odeset('RelTol',1e-4,'AbsTol',1e-6);
```
- Use enough points for the time vector.
- Go over the example from the tutorial, to see how to use `ODE45` properly.

First, consider the single degree of freedom system:



A weightless pendulum with length L and a point mass m at its end is connected to the wall with a joint, where the friction is modeled with an angular viscous damper, c_2. The input to the system is the torque T, and the output is the pendulum's angular position. The system is subjected to gravitation, g.

The system parameters are given as:

$$m = 2.3\,\text{kg}, \quad c_2 = 15\times10^{-3}\,\text{Nms}, \quad L = 0.12\,\text{m}, \quad g = 10\,\text{m/s}^2$$

1) Derive the equation of motion of the mass attached to the pendulum.

2) Find the equilibrium points for T=0.

3) Linearization:
   a. Write the state-space representation of the system.
   b. Linearize the system at the first stable equilibrium point, where the pendulum is at its lowest point.

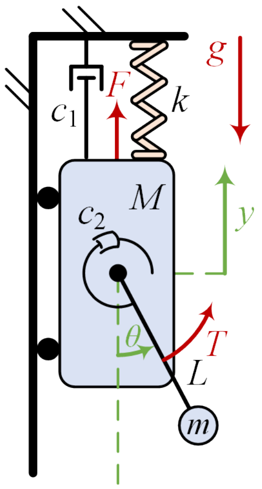
4) Simulate the **linear** and **nonlinear** system for the following initial conditions when $T = 0$.

$$\begin{pmatrix} \theta(0) \\ \dot{\theta}(0) \end{pmatrix} = \left\{ \begin{pmatrix} 2^\circ \\ 0 \end{pmatrix} \quad \begin{pmatrix} 65^\circ \\ 0 \end{pmatrix} \right\}$$

a. Implement the linearized system using the `ss` function in MATLAB (see Example_ss.m), and the nonlinear system should in EX_4_NLode_1DOF_2025.m.

b. Use the `initial` (see Example_ss.m) functions to simulate the linearized system and `ode45` for the nonlinear system. Plot the angle over time as well as the difference between the linear and nonlinear solution. What can you say about the use of the linear model compared to the nonlinear one?

Next, consider the modified system with two degrees of freedom:



The system now comprises a cart with mass M that is attached to the wall with a linear spring k and a viscous damper c_1. The weightless pendulum of length L and point mass m at its end is connected to the cart with a joint, where the friction is modeled with an angular viscous damper, c_2. The inputs to the system are the force F and the torque T, and the outputs are the cart's position and the pendulum's angular position. The system is subjected to gravitation, g.

The system parameters are given as:

$$M = 3.1\text{kg}, \quad m = 2.3\text{kg}, \quad k = 2150\text{N/m}, \quad c_1 = 13.2\text{Ns/m}, \quad c_2 = 15\times10^{-3}\text{ Nms}, \quad L = 0.12\text{m}, \quad g = 10\,\text{m/s}^2.$$

5) Derive the kinetic and potential energies and the Rayleigh dissipation function. Start by writing the position vectors of m and M.

The **nonlinear** governing equations of motion that can be derived from the Euler-Lagrange equations are:

$$\begin{cases} (m+M)\ddot{y} + Lm\sin(\theta)\ddot{\theta} + c_1\dot{y} + Lm\cos(\theta)\dot{\theta}^2 + ky + (m+M)g = F \\ mL^2\ddot{\theta} + mL\sin(\theta)\ddot{y} + c_2\dot{\theta} + mgL\sin(\theta) = T \end{cases}$$

$$\begin{cases} \ddot{y} = -g + 2\dfrac{(c_2\dot{\theta} - T)\sin(\theta) - L\left[c_1\dot{y} - F + ky + \dot{\theta}^2 Lm\cos(\theta)\right]}{L\left[m + 2M + m\cos(2\theta)\right]} \\ \ddot{\theta} = 2\dfrac{Lm\left(c_1\dot{y} - F + ky + \dot{\theta}^2 Lm\cos(\theta)\right)\sin(\theta) - (m+M)(c_2\dot{\theta} - T)}{L^2 m\left[m + 2M + m\cos(2\theta)\right]} \end{cases} \tag{1}$$

6) Find the equilibrium positions for $F = T = 0$.

7) Sort the equilibrium points into stable and unstable points:
   a. Plot the potential energy for $-2(m+M)g/k<y<0$, and $1.5\pi<\theta<1.5\pi$.
      - What are the red points highlighted?
      - Can you say if there are any equilibrium points? Which ones are stable? Which unstable?

   b. Derive the Hessian of the potential energy

$$H\{V(y,\theta)\} = \begin{pmatrix} \frac{\partial^2 V}{\partial y^2} & \frac{\partial^2 V}{\partial y \partial \theta} \\ \frac{\partial^2 V}{\partial \theta \partial y} & \frac{\partial^2 V}{\partial \theta^2} \end{pmatrix}$$

   If the matrix at certain coordinates is positive definite (i.e., all eigenvalues are positive) the point is positive definite. You can use the `eig` function in MATLAB.
      - What can you tell about the stability of the equilibrium points with the help of the Hessian?

The state-space representation for the system is given by:

$$\mathbf{x}^{\mathrm{T}} = \begin{pmatrix} y & \theta & \dot{y} & \dot{\theta} \end{pmatrix}, \quad \mathbf{u}^{T} = \begin{pmatrix} F & T \end{pmatrix}, \quad \mathbf{y}^{T} = \begin{pmatrix} y & \theta \end{pmatrix}$$

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = -g + 2\frac{(c_2 x_4 - T)\sin(x_2) - L\left[c_1 x_3 - F + kx_1 + x_4^2 Lm\cos(x_2)\right]}{L\left[m + 2M + m\cos(2x_2)\right]}$$

$$\dot{x}_4 = 2\frac{Lm\left(c_1 x_3 - F + kx_1 + x_4^2 Lm\cos(x_2)\right)\sin(x_2) - (m+M)(c_2 x_4 - T)}{L^2 m\left[m + 2M + m\cos(2x_2)\right]}$$

$$y_1 = x_1$$

$$y_2 = x_2$$

By linearizing the system near the first equilibrium point at which the pendulum is in the **lowest position** we obtain the following linear state-space representation:

$$\delta\dot{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m+M} & 0 & -\frac{c_1}{m+M} & 0 \\ 0 & -\frac{g}{L} & 0 & -\frac{c_2}{L^2 m} \end{pmatrix}\delta\mathbf{x} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m+M} & 0 \\ 0 & \frac{1}{L^2 m} \end{pmatrix}\delta\mathbf{u}$$

$$\delta\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\delta\mathbf{x}$$

Where $\delta\boldsymbol{x} = \boldsymbol{x} - \overline{\boldsymbol{x}}$, $\delta\boldsymbol{u} = \boldsymbol{u} - \overline{\boldsymbol{u}}$, and $\delta\boldsymbol{y} = \boldsymbol{y} - \overline{\boldsymbol{y}}$

8) Simulate the **linear** and **nonlinear** system for the following initial conditions when F=T = 0, and $\bar{y}$ is the value of y at the stable equilibrium point. The nonlinear system is already implemented in EX_4_NLode_2025.m.

Notice that the linear solution is given **centered around the equilibrium point ($\delta\mathbf{y}$)** whereas the nonlinear solution is in the original coordinates ($\mathbf{y}$). Plot the position $\delta\mathbf{y} = \mathbf{y} - \bar{\mathbf{y}}$, and the angle θ over time, as well as the difference between the linear and nonlinear solutions. What can you say about the linear model compared to the nonlinear one?

$$\begin{pmatrix} y(0) \\ \theta(0) \\ \dot{y}(0) \\ \dot{\theta}(0) \end{pmatrix} = \left\{ \begin{pmatrix} \bar{y} \\ 2^\circ \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1.1\bar{y} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ 65^\circ \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 4\bar{y} \\ 2^\circ \\ 0 \\ 0 \end{pmatrix} \right\} \Leftrightarrow \begin{pmatrix} \delta y(0) \\ \delta\theta(0) \\ \delta\dot{y}(0) \\ \delta\dot{\theta}(0) \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ 2^\circ \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0.1\bar{y} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 65^\circ \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 3\bar{y} \\ 2^\circ \\ 0 \\ 0 \end{pmatrix} \right\}$$

9) Simulate the **linear** and **nonlinear** system response to T = 0, and $F = 28\sin(\omega t)$, where the initial conditions are:

$$\begin{pmatrix} y(0) \\ \theta(0) \\ \dot{y}(0) \\ \dot{\theta}(0) \end{pmatrix} = \begin{pmatrix} \bar{y} \\ 2^\circ \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \delta y(0) \\ \delta\theta(0) \\ \delta\dot{y}(0) \\ \delta\dot{\theta}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2^\circ \\ 0 \\ 0 \end{pmatrix}$$

Use the four different frequencies: ω = {3, 22, 41} rad/s.
Use the `ss` and `lsim` (see Example_ss.m) functions for the linear system and `ode45` for the nonlinear system.
Explain what happens to the system (you can inspect the plot in MATLAB to better understand what happens). Does the linear model generate similar results? What can you say about the oscillatory frequency of the system?

10) Explore the **linear** and **nonlinear** system response to an input step in F and T.
   a. We define the input amplitude as $F = A_F$ for $0 \le t$. The initial conditions are given as the stable equilibrium point found in 6). Run simulations for AF = {6, 190} N. (T=0).
   b. We define the input amplitude as $T = A_T$ for $0 \le t$. The initial conditions are the same as in 10a, Run simulations for A_T = {0.15, 1.35} Nm. (F=0)

Use the `ss` and `step` (see Example_ss.m) functions for the linear system and `ode45` for the nonlinear system. Describe the cart and pendulum trajectories for each A value according to the nonlinear model, and comment on the responses of the linear vs. nonlinear systems.