Problem 1

- mass m

$$m\ddot{z}_1 = -k_1 z_1 - k_2 (z_1 - z_2)$$

- point O

$$0 = -f\dot{z}_2 - k_2 (z_2 - z_1) - k_3 (z_2 - z_3)$$

- point mass O'

$$0 = F - k_3 (z_3 - z_2)$$

$$m\ddot{z}_1 = -k_1 z_1 - k_2 (z_1 - z_2) \qquad (1)$$
$$0 = -f\dot{z}_2 - k_2 (z_2 - z_1) + F \qquad (2)$$

equations do not depend on k_3.

b) $x_1 = z_1$, $x_2 = \dot{z}_1$, $x_3 = z_2$

input: F output: z_1 $u = F$ and $y = z_1$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}\left\{-(k_1+k_2)x_1 + k_2 x_3\right\}$$
$$\dot{x}_3 = \frac{1}{f}\left[k_2 x_1 - k_2 x_3\right] + \frac{1}{f}u$$

$$y = x_1$$

c) Laplace Transform

(1) $$ms^2 Z_1(s) = -k_1 Z_1(s) - k_2 (Z_1(s) - Z_2(s))$$

(2) $$0 = -fs Z_2(s) - k_2 (Z_2(s) - Z_1(s)) + F(s)$$

$$Z_2(s) = \frac{F(s) + k_2 Z_1(s)}{fs + k_2} \quad \text{from (2)}$$

From (1)

$$Z_1(s)\left[ms^2 + k_1 + k_2 - \frac{k_2^2}{fs+k_2}\right] = F(s)\frac{k_2}{fs+k_2}$$

$$\frac{Z_1(s)}{F(s)} = \frac{k_2}{mfs^3 + mk_2 s^2 + (k_1+k_2)fs + k_1 k_2}$$

d) circuit diagram

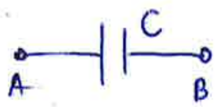
$m \leftrightarrow L$ $\quad Z_1 \leftrightarrow q_1$ $\quad i_1 = \frac{dq_1}{dt}$

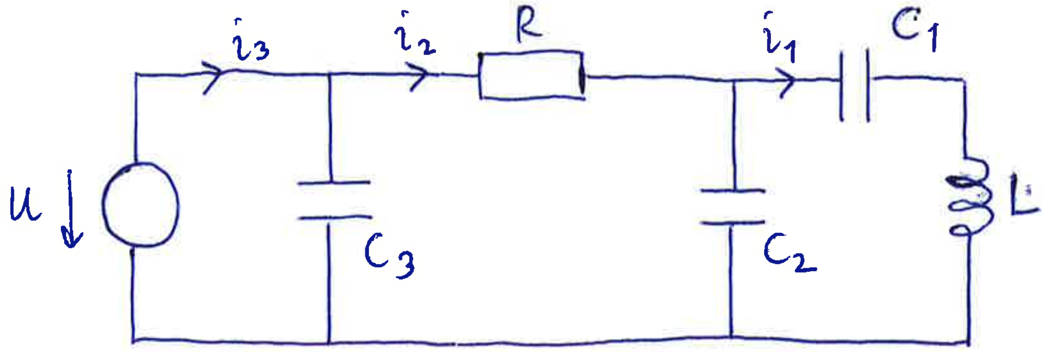
$1/k_1 \leftrightarrow C_1$ $\quad Z_2 \leftrightarrow q_2$

$1/k_2 \leftrightarrow C_2$ $\quad Z_3 \leftrightarrow q_3$

$f \leftrightarrow R$ $\quad F \leftrightarrow u$ $\quad u_{AB} = \frac{q}{C}$

$1/k_3 \leftrightarrow C_3$





Loop A: $u = \frac{1}{C_3} \int (i_3 - i_2)\,dt \rightarrow 0 = u - \frac{1}{C_3}(q_3 - q_2)$

Loop B: $R i_2 + \frac{1}{C_2} \int (i_2 - i_1)\,dt - \frac{1}{C_3}(i_3 - i_2)\,dt = 0$

$$0 = -R\dot{q}_2 - \frac{1}{C_2}(q_2 - q_1) - \frac{1}{C_3}(q_2 - q_3)$$

Loop C: $\frac{1}{C_2} \int (i_2 - i_1)\,dt = L\frac{di_1}{dt} + \frac{1}{C_1} \int i_1\,dt$

$$L\ddot{q}_1 = -\frac{1}{C_1} q_1 - \frac{1}{C_2}(q_1 - q_2)$$

Problem 2

a) Equilibrium point

$$0 = 2\bar{x}_1 + 2\bar{x}_1\bar{x}_2 + \bar{u} \qquad (1)$$

$$0 = \bar{x}_2 + 3\bar{x}_1\bar{x}_2 \qquad (2)$$

$3\cdot(1) - 2\cdot(2)$

$$6\bar{x}_1 + 6\bar{x}_1\bar{x}_2 + 3\bar{u} - 2\bar{x}_2 - 6\bar{x}_1\bar{x}_2 = 0$$

$$6\bar{x}_1 - 2\bar{x}_2 + 3\bar{u} = 0 \rightarrow \bar{x}_2 = 3\bar{x}_1 + 1.5\bar{u}$$

From (1)

$$2\bar{x}_1 + 2\bar{x}_1(3\bar{x}_1 + 1.5\bar{u}) + \bar{u} = 0$$

$$6\bar{x}_1^2 + \bar{x}_1(2 + 3\bar{u}) + \bar{u} = 0$$

For $\bar{u} = 1$ $\quad 6\bar{x}_1^2 + 5\bar{x}_1 + 1 = 0$

$$\bar{x}_1 \rightarrow -1/3, \; -1/2$$

$$\bar{x}_2 = 3\bar{x}_1 + 1.5\bar{u} = \; \rightarrow 1/2, \; 0$$

$\bar{x}_1, \bar{x}_2 \neq 0$ then $\bar{x}_1 = -\frac{1}{3}$ and $\bar{x}_2 = \frac{1}{2}$ $\quad (\bar{u} = 1)$

b) $\dot{x}_1 = f_1(x_1, x_2, u)$

$\dot{x}_2 = f_2(x_1, x_2, u)$

Linearized equations

$$\begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_2 \end{bmatrix} = A \cdot \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + B \delta u$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}, \bar{u}} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2+2\bar{x}_2 & 2\bar{x}_1 \\ 3\bar{x}_2 & 1+3\bar{x}_1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2/3 \\ 3/2 & 0 \end{bmatrix}$$

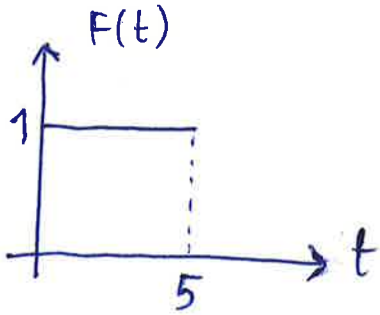
$$B = \left.\frac{\partial f}{\partial u}\right|_{\bar{x}, \bar{u}} = \begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\delta \dot{x}(t) = \begin{bmatrix} 3 & -2/3 \\ 3/2 & 0 \end{bmatrix} \cdot \delta x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta u$$

$$\delta x(0) = x(0) - \bar{x}(0) = \begin{bmatrix} 1/3 \\ -1/2 \end{bmatrix}$$

Problem 3

a)



$$F(t) = \varepsilon(t) - \varepsilon(t-5)$$

$$\downarrow$$

$$F(s) = \frac{1}{s}\left(1 - e^{-5s}\right)$$

$$ms^2 X(s) = F(s) - kX(s) - fsX(s)$$

$$(ms^2 + fs + k)X(s) = F(s)$$

$$X(s) = \frac{F(s)}{ms^2 + fs + k}$$

$$= \frac{1}{s(ms^2+fs+k)} - \frac{e^{-5s}}{s(ms^2+fs+k)}$$

$$= \underbrace{\frac{1}{s}\cdot\frac{0.5}{s^2+4s+5}}_{X_1(s)} - \frac{1}{s}\cdot\frac{0.5}{s^2+4s+5}\cdot e^{-5s}$$

$$X_1(s) = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5}$$

$$A = \lim_{s\to 0} s X_1(s) = 0.1$$

$$0.1(s^2+4s+5) + (Bs+C)s = 0.5$$

$$(0.1+B)s^2 + (0.4+C)s + 0.5 = 0.5$$

$$B = -0.1 \text{ and } C = -0.4$$

$$X_1(s) = \frac{0.1}{s} - \frac{0.1s+0.4}{s^2+4s+5}$$

$$= \frac{0.1}{s} - 0.1\left[\frac{s+2}{(s+2)^2+1} + 2\cdot\frac{1}{(s+2)^2+1}\right]$$

$$x_1(t) = \left(0.1 - 0.1e^{-2t}\cos t - 0.2e^{-2t}\sin t\right)\varepsilon(t)$$

$$x(t) = x_1(t) + \left(0.1 - 0.1e^{-2(t-5)}\cos(t-5) - 0.2e^{-2(t-5)}\sin(t-5)\right)\varepsilon(t-5)$$

b) $U(s) = \frac{1}{s}$

$$G(s) = \frac{1}{s+1} + \frac{1}{s+2}$$

$$Y(s) = \underbrace{\frac{1}{s(s+1)}}_{Y_1(s)} + \underbrace{\frac{1}{s(s+2)}}_{Y_2(s)}$$

$$Y_1(s) = \frac{A_1}{s} + \frac{B_1}{s+1}$$

$$A_1 = \lim_{s\to 0}\frac{1}{s+1} = 1$$

$$B_1 = \lim_{s\to -1}\frac{1}{s} = -1$$

$$Y_2(s) = \frac{A_2}{s} + \frac{B_2}{s+2}$$

$$A_2 = \lim_{s\to 0}\frac{1}{s+2} = 1/2$$

$$B_2 = \lim_{s\to -2}\frac{1}{s} = -1/2$$

$$y(t) = \left[1 - e^{-t} + \frac{1}{2} - \frac{1}{2}e^{-2t}\right]\varepsilon(t) = \left[1.5 - e^{-t} - 0.5e^{-2t}\right]\varepsilon(t)$$

Convolution

$$y(t) = \varepsilon(t) * g(t) = \int_0^t \varepsilon(\tau)\, g(t-\tau)\, d\tau$$

For $t<0$ $\quad \int_0^t 0 \, d\tau = 0$

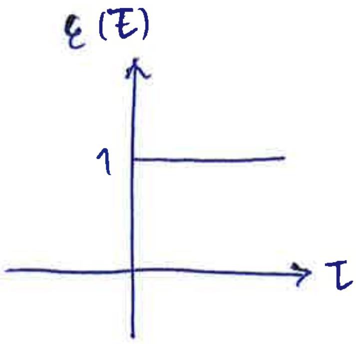
$t \geqslant 0 \quad \int_0^t e^{-(t-\tau)} d\tau + \int_0^t e^{-2(t-\tau)} d\tau$

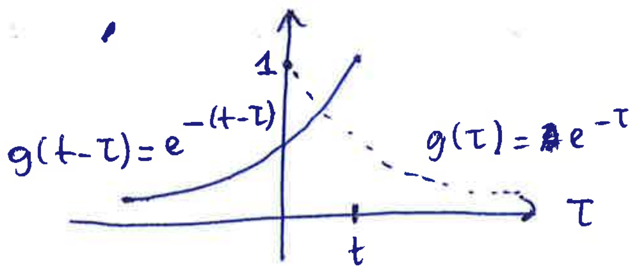
$$= e^{-t} \int_0^t e^{\tau} d\tau + e^{-2t} \int_0^t e^{2\tau} d\tau$$

$$= e^{-t}\left(e^{t}-1\right) + e^{-2t}\left(\frac{1}{2} e^{2t} - \frac{1}{2}\right)$$

$$= 1 - e^{-t} + \frac{1}{2} - \frac{1}{2} e^{-2t}$$

$$= 1.5 - e^{-t} - \frac{1}{2} e^{-2t} \qquad t \geqslant 0$$





c) $\dot{y}(t) \longrightarrow sY(s) - y(0) = sY(s) - 2$

$\ddot{y}(t) \rightarrow s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s) - 2s - 1$

$\dddot{y}(t) \rightarrow s^3Y(s) - s^2 y(0) - s\dot{y}(0) - \ddot{y}(0)$

$= s^3 Y(s) - 2s^2 - s - 0.5$

$$Y(s) = \frac{2s^2 + 17s + 42.5}{s^3 + 8s^2 + 17s + 10}$$

initial value theorem

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} \frac{2s^3 + 17s^2 + 42.5s}{s^3 + 8s^2 + 17s + 10} = 2$$

$$Y(s) = \frac{2s^2 + 17s + 42.5}{(s+1)(s+2)(s+5)}$$

note that $s(s^2 + 8s + 15) + 2s + 10$
$\downarrow$
$s(s+3)(s+5) + 2(s+5)$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+5}$$

$$A = \lim_{s \to -1} (s+1)Y(s) = 27.5/4$$

$$y(t) = \frac{27.5}{4} e^{-t} - \frac{16.5}{3} e^{-2t} + \frac{2.5}{4} e^{-5t} \quad t \geq 0$$

Problem 4

a) $G(s) = \dfrac{K}{s} \dfrac{1}{(\tau s+1)^2}$ $\quad K = 2$, $\tau = 1$

order = 3 $\quad$ repeated pole at -1 and a pole at origin

static gain $\to \infty$
(DC)

b) $F(s) = \dfrac{K_f(s+1)}{\tau_f s + 1}$

$$G_f(s) = \frac{K_f(s+1)}{(\tau_f s+1)} \cdot \frac{2}{s(s+1)^2} = \frac{2K_f}{(\tau_f s+1)s(s+1)}$$

$$|G_f(j\omega)| = \frac{2K_f}{\sqrt{1+\tau_f^2\omega^2}\,\omega\sqrt{1+\omega^2}}$$

$$\angle G_f(j\omega) = -\frac{\pi}{2} - \arctan(\tau_f \omega) - \arctan(\omega)$$

at $\omega = 1$ $\quad |G_f(j\omega)| = \dfrac{2K_f}{\sqrt{1+\tau_f^2}\sqrt{2}} = 1$

$$\varphi = -90° - \arctan(\tau_f) - \arctan(1) = -145°$$

$$\arctan(\tau_f) = 10° \to \tau_f = 0.176$$

$$K_f = \sqrt{\frac{1+\tau_f^2}{2}} = 0.718$$

$$F(s) = \frac{0.718(s+1)}{0.176s+1}$$