Solution

1. To derive the governing equations of motion we first write the radius vector to each mass.

$$\mathbf{r}_M = x\hat{\mathbf{x}}, \quad \mathbf{r}_m = \left[x + L\sin(\theta)\right]\hat{\mathbf{x}} - L\cos(\theta)\hat{\mathbf{y}} \tag{1}$$

The velocities are:

$$\dot{\mathbf{r}}_M = \dot{x}\hat{\mathbf{x}}, \quad \dot{\mathbf{r}}_m = \left[\dot{x} + L\dot{\theta}\cos(\theta)\right]\hat{\mathbf{x}} + L\dot{\theta}\sin(\theta)\hat{\mathbf{y}} \tag{2}$$

From that, we obtain:

$$\left|\dot{\mathbf{r}}_M\right|^2 = \dot{x}^2, \quad \left|\dot{\mathbf{r}}_m\right|^2 = \dot{x}^2 + L^2\dot{\theta}^2 + 2L\cos(\theta)\dot{x}\dot{\theta} \tag{3}$$

The kinetic and potential energies, Rayleigh dissipation function, and the Lagrangian are

$$\begin{aligned}
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + L^2\dot{\theta}^2 + 2L\cos(\theta)\dot{x}\dot{\theta}\right] \\
V &= \frac{1}{2}kx^2 - mgL\cos(\theta) \\
D &= \frac{1}{2}c_1\dot{x}^2 + \frac{1}{2}c_2\dot{\theta}^2 \\
\mathcal{L} &= T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}mL^2\dot{\theta}^2 + mL\cos(\theta)\dot{x}\dot{\theta} - \frac{1}{2}kx^2 + mgL\cos(\theta)
\end{aligned} \tag{4}$$

2. To derive the state-space representation we begin with defying the state vector, and input vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{pmatrix} \Rightarrow \dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} F & T \end{pmatrix}^T$$

Then, from the equations of motion, we have:

$$\begin{aligned}
\dot{x}_3 &= -\frac{c_1}{M}x_3 + \frac{c_2}{LM}x_4 - \frac{k}{M}x_1 + \frac{mg}{M}x_2 + \frac{u_1}{M} - \frac{u_2}{LM} \\
\dot{x}_4 &= -\frac{(m+M)c_2}{L^2Mm}x_4 + \frac{c_1}{LM}x_3 - \frac{(m+M)g}{LM}x_2 + \frac{k}{LM}x_1 - \frac{u_1}{LM} + \frac{M+m}{L^2Mm}u_2
\end{aligned}$$

Recasting into matrix form including the output we obtain:

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{M} & \frac{mg}{M} & -\frac{c_1}{M} & \frac{c_2}{LM} \\ \frac{k}{LM} & -\frac{(m+M)g}{LM} & \frac{c_1}{LM} & -\frac{(m+M)c_2}{L^2Mm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{M} & -\frac{1}{LM} \\ -\frac{1}{LM} & \frac{M+m}{L^2Mm} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & L \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{c_1 L} \\ 0 & 0 \end{pmatrix} \mathbf{u}$$

3. The transfer function is obtained as follows, where T=0 sine it is not an input.

$$\ddot{x}+8\dot{x}-4\dot{\theta}+200x-20\theta=2F$$
$$\ddot{\theta}+12\dot{\theta}-16\dot{x}+60\theta-400x=-4F$$

$$\left(s^2+8s+200\right)X-\left(4s+20\right)\Theta=2\tilde{F}$$
$$\left(s^2+12s+60\right)\Theta-\left(16s+400\right)X=-4\tilde{F}$$

$$+\begin{cases}-\left(s^2+12s+60\right)\left(4s+20\right)\Theta+\left(s^2+12s+60\right)\left(s^2+8s+200\right)X=2\left(s^2+12s+60\right)\tilde{F}\\ \left(s^2+12s+60\right)\left(4s+20\right)\Theta-\left(4s+20\right)\left(16s+400\right)X=-4\left(4s+20\right)\tilde{F}\end{cases}$$

$$\left[\left(s^2+12s+60\right)\left(s^2+8s+200\right)-\left(4s+20\right)\left(16s+400\right)\right]X=\left[2\left(s^2+12s+60\right)-4\left(4s+20\right)\right]\tilde{F}$$

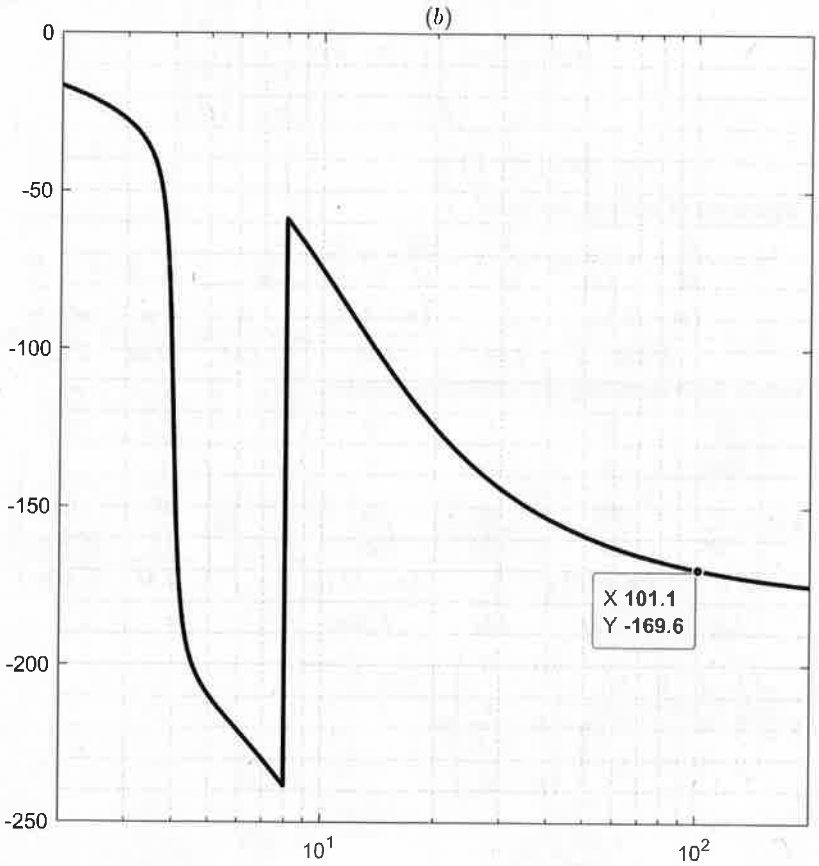
$$G(s)=\frac{X}{\tilde{F}}=\frac{2\left(s^2+12s+60\right)-4\left(4s+20\right)}{\left(s^2+12s+60\right)\left(s^2+8s+200\right)-\left(4s+20\right)\left(16s+400\right)}=\frac{2s^2+8s+40}{s^4+20s^3+292s^2+960s+4000}$$

4. Analyzing the numerator, we can find that we have a complex zero $z_{1,2}=\pm 8i$, suggesting a strong dip in the Bode plot at 8 rad/s. From the denominator we can find:

$$\omega_1=12\,\text{rad/s},\qquad \zeta_1=0.75>0.707$$
$$\omega_2=4\,\text{rad/s},\qquad \zeta_2=0.03<0.707$$

This suggest a resonance peak at 4 rad/s and overdamped response, i.e., no resonance peak at 12 rad/s. Therefore, the correct plot is b.

5. Since the transfer function has four complex poles (-2π) and two real zeroes (+π), the phase after which should be –π (-180°).



Problem 2

$$G(s) = \frac{b_0 s}{s + a_1}$$

1. $|G(jw)| = 5$ for $w >> a_1$

$$G(jw) = \frac{b_0 (jw)}{jw + a_1} \qquad |G(jw)| = \frac{b_0 w}{\sqrt{w^2 + a_1^2}}$$

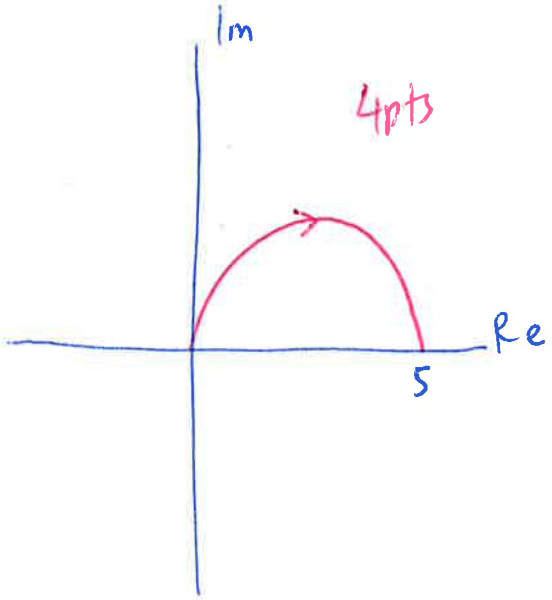
for $w >> a_1$ $\quad |G(jw)| = b_0 \quad b_0 = 5 \quad$ 3pts

$\angle G(jw) = \pi/4$ for $w = 10$ rad/sec corner frequency

$a_1 = 10 \quad$ 2pts $\qquad G(s) = \dfrac{5s}{s+10}$

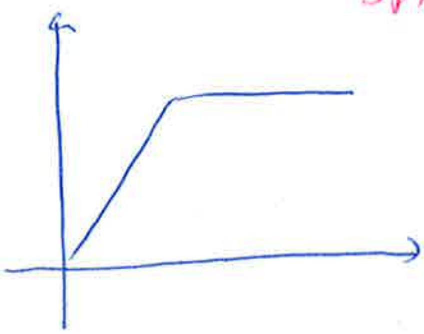
2.

4pts



3. high-pass

3pts



Problem 3

1. $$G(s) = \frac{K(s-z_1)}{(s-p_1)(s-p_2)}$$ 3pts

2. $$G(s) = \frac{2(s+5)}{(s+2)(s+1)} \qquad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s}\,\frac{2(s+5)}{(s+2)(s+1)}$$

$$= \frac{A_1}{s} + \frac{A_2}{(s+2)} + \frac{A_3}{(s+1)}$$

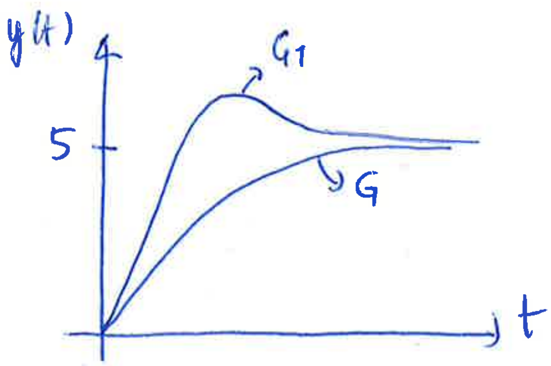
$A_1 = 5,\quad A_2 = 3,\quad A_3 = -8$

$$y(t) = (5 + 3e^{-2t} - 8e^{-t})\,\varepsilon(t)$$ 4pts

3. $$G_1(s) = \frac{2(s+0.5)}{(s+2)(s+1)} \qquad y(t) = (0.5 + e^{-t} - 1.5e^{-2t})\,\varepsilon(t)$$

overshoot

6pts



Problem 4

$$G(s) = \frac{3s}{(s^2+1)^2}$$

1. $G(s) = -\frac{3}{2}\frac{d}{ds}\frac{1}{s^2+1}$ 2pts

$$g(t) = -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{d}{ds}\frac{1}{s^2+1}\right\}$$

$$= \frac{3}{2}t\sin t$$ 2pts

2. $G(s) = F(s)\cdot H(s)$ where $F(s) = \frac{3}{s^2+1}$, $H(s) = \frac{s}{s^2+1}$ 2pts

$f(t) = 3\sin t$, $h(t) = \cos t$ 2pts

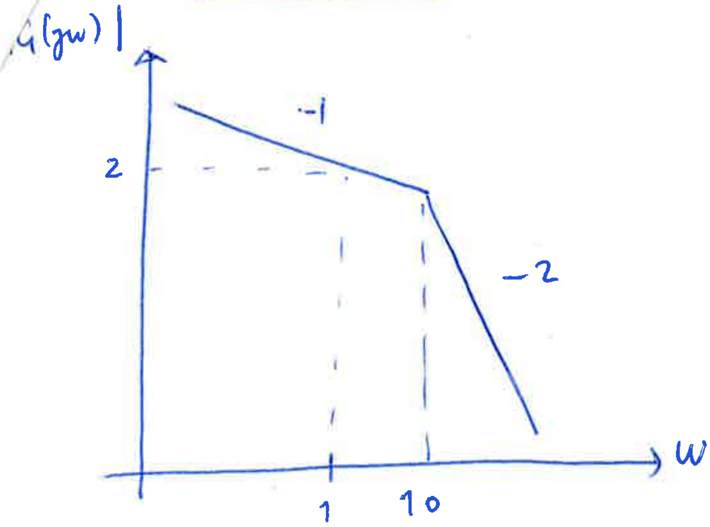
$$g(t) = \mathcal{L}^{-1}\{F(s)H(s)\} = (f * h)(t)$$

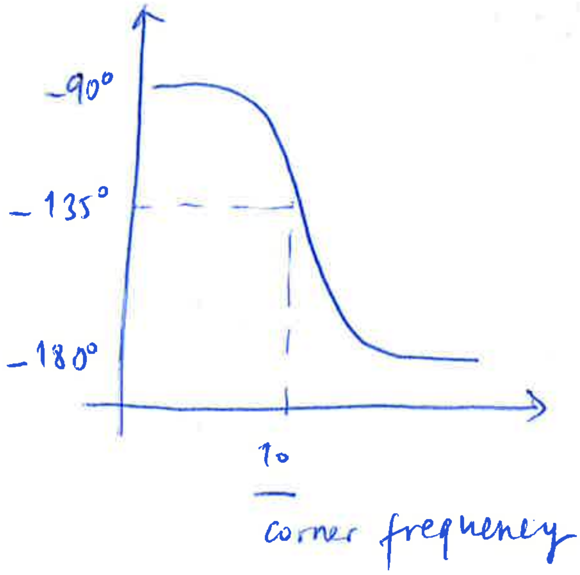
$$= 3\int_0^t \sin(t-v)\cos v\, dv$$ 1pt

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$ 2pts

$$g(t) = \frac{3}{2}\int_0^t \sin t\, dv + \int_0^t \sin(t-2v)\, dv = \frac{3}{2}t\sin t$$

Problem 5





2pts

$$G(s) = \frac{A}{s(s+10)}$$

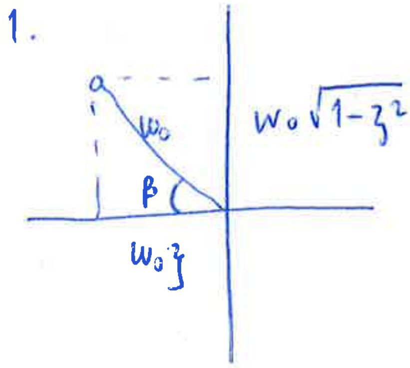
3pts 3pts

$$A = 20 \rightarrow G(s) = \frac{2}{s\left(\frac{s}{10}+1\right)}$$

2. phase would decrease indefinitely 2pts

Problem 6

1.



$$20 < w_0 < 30$$ 2pts

$$\cos(60°) < \zeta < \cos(30°)$$

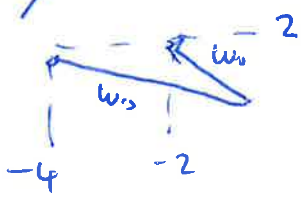
$$\frac{1}{2} < \zeta < \frac{\sqrt{3}}{2}$$ 3pts

1pts

2. same frequency of oscillations $\left(\bar{w} = w_0\sqrt{1-\zeta^2}\right)$

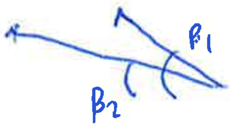
same peak time $\left(t_p = \frac{\pi}{\bar{w}}\right)$ 1pts

lower settling time $\left(t_s = \frac{4}{\zeta w_0}\right)$ 1pts



1pts lower rise time $\left(t_r = \frac{1.8}{w_0}\right)$

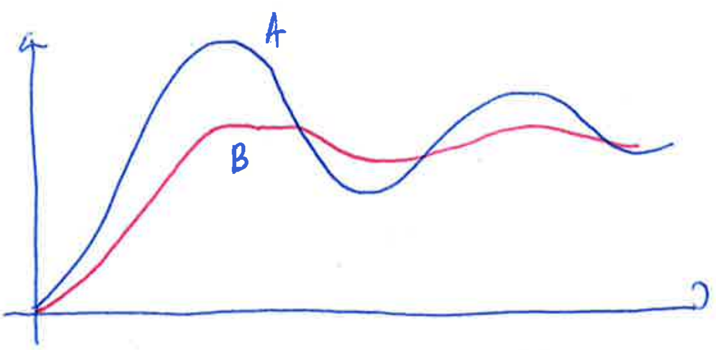
as w_0 increases



$\zeta = \cos(\beta)$

1pts lower overshoot

as β decreases



3.

(1) $\dfrac{25}{s^2+6s+25}$ $\quad w_0 = 5 \quad \zeta = 0.6 \quad$ underdamped w resonance

2pts 2pts

(2) $\dfrac{16}{s^2+6s+16}$ $\quad w_0 = 4 \quad \zeta = 0.75 \quad$ underdamped w/o resonance

(3) $\dfrac{8}{s^2+6s+8}$ $\quad w_0 = 2\sqrt{2} \quad \zeta = \dfrac{3\sqrt{2}}{4} \quad$ overdamped 2pts

From (1) to (2)

overshoot decreases 1pts

rise time and peak time increases 2pts →slower response

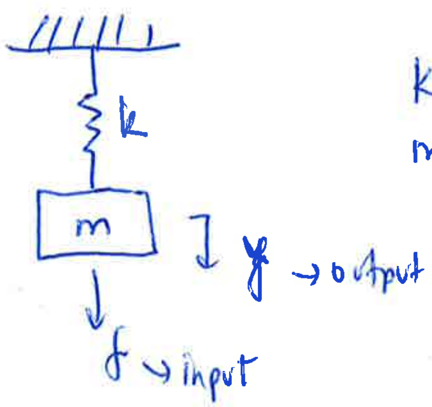
same settling time 1pts

4. $G(s) = \dfrac{1}{2s^2+3}$ $\quad \zeta = 0$ 1pt

$2s^2Y(s) + 3Y(s) = U(s)$ ~~2pts~~

$$2\frac{d^2y(t)}{dt^2} + 3y(t) = u(t)$$

3pts



1pt

$k = 3$

$m = 2$