ME-221 Midterm Exam Solutions —— Spring 2019

Problem 1 (35 points)

1. (5 points) To analyze the system, we begin with the electrical circuit. The back-emf (the voltage induced by the rotation of the motor) is given by $E_m(t) = K_m\omega(t)$.

$$u(t) = R_m i(t) + K_m\omega(t) \tag{1}$$

Next, we must write the equation of motion as torque balance on the motor shaft. The motor's torque is provided by the electrical current $i(t)$ through the windings and given by $T_m = K_t i(t)$. There are three torques counteracting the motor: (i) the inertia of the motor (ii) the rotational viscous damper f_R and (iii) the torque exercised by the translational elements (mass M and translational viscous damper f_T) via the pinion gear. Let's denote the last term by T_p.

$$K_t i(t) = J\dot{\omega}(t) + f_R\omega(t) + T_p(t) \tag{2}$$

Next, we write down the force-balance on the translational part of the load.

$$F_p(t) = M\dot{v}(t) + f_T v(t) \tag{3}$$

Lastly, assuming that the pinion is perfectly cylindrical and neglecting backlash or other non-linearities due to gears, we have $T_p(t) = rF_p(t)$ and $v(t) = r\omega(t)$.

2. (10 points) We will take the Laplace transform of the equations of motion and replace $\Omega(s)$ with $V(s)/r$ and $T_p(s)$ with $rF_p(s)$ as we want to find a relationship between the input $U(s)$ and the output $V(s)$. From (2) and (3) we get:

$$K_t I(s) = (Js + f_R)\frac{V(s)}{r} + r(Ms + f_T)V(s) \rightarrow I(s) = (\frac{J + r^2 M}{rK_t}s + \frac{f_R + r^2 f_T}{rK_t})V(s) \tag{4}$$

Entering $I(s)$ as a function of $V(s)$ into (1) results in the transfer function as follows:

$$U(s) = R(\frac{J + r^2 M}{rK_t}s + \frac{f_R + r^2 f_T}{rK_t})V(s) + K_m\frac{V(s)}{r} \rightarrow G(s) = \frac{V(s)}{U(s)} = \frac{\frac{rK_t}{R(J+r^2 M)}}{s + \frac{f_R + r^2 f_T + K_m K_t/R}{J + r^2 M}} \tag{5}$$

The order of the system is 1.

3. (10 points) Substituting the numerical values of the parameters, we obtain $G(s) = \frac{0.25}{s+3}$. The steady-state (DC) gain K is $1/12$ and the time constant τ is $1/3$. The output reaches 98% of its steady-state value in $4\tau = 4/3$. The unit-step response is given by $y(t) = 1/12(1 - e^{-3t})$.

4. (10 points) $Y(s) = 4\frac{e^{-3s}}{s+2}\frac{0.25}{s+3} = \frac{e^{-3s}}{(s+2)(s+3)}$. By taking the inverse Laplace transform of $\frac{1}{(s+2)(s+3)}$ which is given by $y(t) = e^{-2t} - e^{-3t}$ and shifting the response by 3 due to the exponential term we obtain $y(t) = e^{-2(t-3)} - e^{-3(t-3)}$.

Problem 2 (40 points)

1. (10 points) The system is undamped (i.e. $\zeta = 0$) so $y(t) = K(1 - cos(\omega_0 t))$. From the plot we can read that $K = 2$ and $\omega_0 = 3$.

2. (5 points) The poles, which are complex conjugates, are given by $p_{1,2} = \pm j3$. The system is unstable due to undamped oscillations. If we move the poles to the left, the system will become stable. If we move the poles to the right, we will have an unstable system with growing oscillations.

3. (10 points) The poles of the new system are $p_{1,2} = -4 \pm j3 = -\omega_0(\zeta \pm \sqrt{\zeta^2 - 1})$. The new system is underdamped and the damping ratio is $\zeta = 0.8$ while the undamped natural frequency is shifted to $\omega_0 = 5$. The new transfer function is given by $G(s) = K\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{50}{s^2 + 8s + 25}$. The rise time, peak time, maximum overshoot, and settling time can be calculated using the equations given in Lecture 8.

4. (15 points) The transfer function of the second-order spring-mass-damper system is given by $G(s) = \frac{1/m}{s^2 + \frac{b}{m}s + \frac{k}{m}}$. We can find the values of the parameters from the transfer function found in part 3 as follows; $m = 1/50$, $k = 0.5$ and $b = 0.16$. Overshoot is given by $M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$. To increase overshoot, we need to decrease the damping ratio. Damping ratio depends on both b and m. We want to keep the natural frequency constant so we cannot change the mass. We must decrease the value of the viscous damping coefficient. Settling time, $\frac{4}{\zeta\omega_0}$, will increase.

Problem 3 (25 points)

1. (5 points) The Laplace transform of the equation is $s^2Y(s) - 2s + 3sY(s) - 6 + 2Y(s) = 3sU(s) + 2U(s)$. The input is the unit-step function ($U(s) = 1/s$) so the Laplace transform of the output is $Y(s) = \frac{3s+2}{s^2+3s+2} \times \frac{1}{s} - \frac{2s+6}{s^2+3s+2} = \frac{3s+2}{(s+2)(s+1)} \times \frac{1}{s} - \frac{2s+6}{(s+2)(s+1)}$. The inverse Laplace transform is $y(t) = (1 + e^{-t} - 2e^{-2t}) + (4e^{-t} - 2e^{-2t}) = 1 + 5e^{-t} - 4e^{-2t}$.

2. (10 points) See Exercise 4 on page 123 in the polycopie

3. (10 points) The order of the system is 3 thus we need 3 state variables. The variables are defined as $x_1 = \omega$, $x_2 = r$, and $x_3 = \dot{r}$. The state equations are:

$$\dot{x_3} = x_1^2 x_2 - 10x_2 + 1 = f_3(x_1, x_2, x_3)$$

$$\dot{x_1} = \frac{u - 2x_1}{2x_2^2} = f_1(x_1, x_2, x_3)$$

$$\dot{x_2} = x_3 = f_2(x_1, x_2, x_3)$$

At the equilibrium point the derivatives $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$. It is given that $\bar{\omega} = 3$, we can then find the equilibrium values of the input and state variables as $\bar{u} = 6$, $\bar{x}_1 = 3$, $\bar{x}_2 = 1$, and $\bar{x}_3 = 0$. Linearization is done by taking the partial derivatives at the equilibrium point as follows:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\bar{x}_2^2} & \frac{2\bar{x}_1 - \bar{u}}{\bar{x}_2^3} & 0 \\ 0 & 0 & 1 \\ 2\bar{x}_1\bar{x}_2 & \bar{x}_1^2 - 10 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\bar{x}_2^2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$