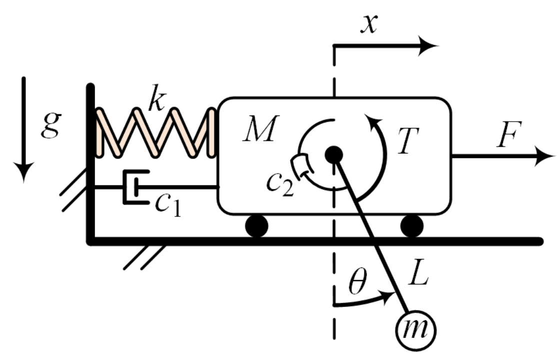
ME-221 Final Exam —— Spring 2022

Problem 1 (30 points)

Consider the mechanical system shown on the right. A cart of mass M is attached to the wall using a linear spring k and a viscous damper c_1. A pendulum of length L carrying a point mass m at its end is connected to the cart through a joint, where the friction at this joint is modelled with a torsional viscous damper c_2. The system is subjected to gravitational acceleration g. The cart is moving on a frictionless surface. The system is initially at rest.



1. (6 points) Write down the potential and kinetic energies, Rayleigh dissipation function, and the Lagrangian of the system. DO NOT derive the equations of motion.

2. (8 points) The linearized equations of motion around the equilibrium point $(x, \theta) = (0, 0)$ are:

$$\begin{aligned} \ddot{x} + \frac{c_1}{M}\dot{x} - \frac{c_2}{LM}\dot{\theta} + \frac{k}{M}x - \frac{mg}{M}\theta &= \frac{F}{M} - \frac{T}{LM} \\ \ddot{\theta} + \frac{(m+M)c_2}{L^2Mm}\dot{\theta} - \frac{c_1}{LM}\dot{x} + \frac{(m+M)g}{LM}\theta - \frac{k}{LM}x &= -\frac{F}{LM} + \frac{M+m}{L^2Mm}T \end{aligned}$$

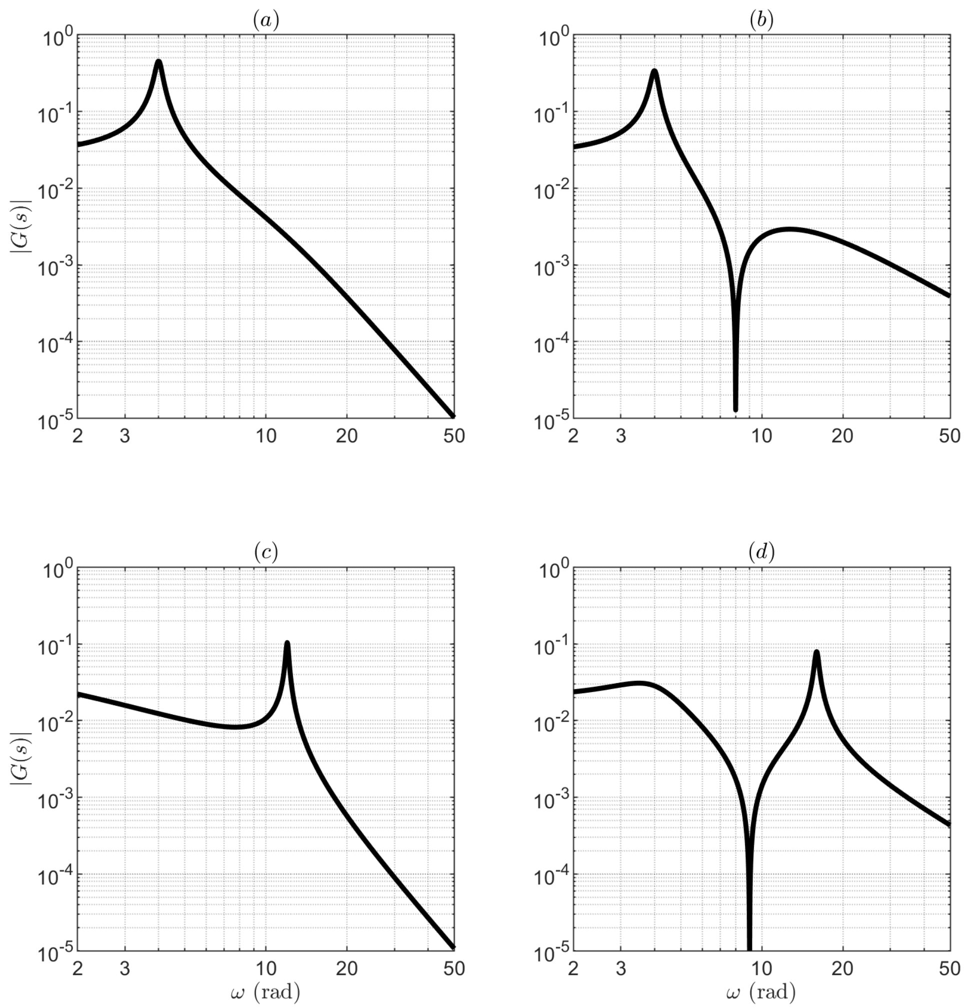
   Derive the state-space representation of the system given that the two inputs to the system are the force F and torque T, and the three outputs are selected as $x + L\theta$, $\dot{x} + \frac{T}{c_1 L}$, and $L\dot{\theta}$.

3. (5 points) The values of the parameters are given as follows:
   $M = \frac{1}{2}$ kg, $m = 1$ kg, $k = 100$ Nm^{-1}, $c_1 = 4$ Nsm^{-1}, $c_2 = 1$ Nsm, $L = \frac{1}{2}$ m, and $g = 10$ ms^{-2}.
   Find the transfer function $G(s) = X(s)/F(s)$ assuming that $T = 0$ (Hint: Do not use the state-space representation).

4. (8 points) For a different set of parameters, the transfer function of the system with T as the input and θ as the output (i.e. $G(s) = Q(s)/T(s)$), is calculated as:

$$G(s) = \frac{s^2 + 64}{(s^2 + 18s + 144)(s^2 + 0.24s + 16)}$$

   Which Bode plot shown on the next page belongs to this transfer function? Justify your result.



5. (3 points) **Estimate** (do not calculate) the phase of the steady-state output for the input $T = 5\sin(100t)$ using the transfer function given in the previous part of the question.

Problem 2 (12 points)

The transfer function of an LTI system is

$$G(s) = \frac{b_0}{1 + a_1 s^{-1}}$$

1. (6 points) Calculate the coefficients b_0 and a_1 such that the frequency response $G(j\omega)$ of the filter satisfies the following criteria: $|G(j\omega)| = 5$ for $\omega >> a_1$ and the phase of $G(j\omega)$ is $\pi/4$ for $\omega = 10$ rad/sec.

2. (4 points) Sketch the Nyquist plot of the system for the calculated values of b_0 and a_1.

3. (2 points) Would this system serve as a high-pass, low-pass, or band-pass filter?

Problem 3 (13 points)

Consider a second order LTI system with two poles p_1 and p_2, one zero z_1, and gain K.

1. (3 points) Write down the transfer function of the system.

2. (4 points) Calculate the unit step response of the system given that $p_1 = -2$, $p_2 = -1$, $z_1 = -5$. and $K = 2$.

3. (6 points) How would the unit step response change if we move the zero from $z_1 = -5$ to $z_1 = -0.5$? Show your work with plots.

Problem 4 (10 points)

Compute the inverse Laplace transform of

$$G(s) = \frac{3s}{(s^2 + 1)^2}$$

1. (4 points) Using the following property: $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$.

2. (6 points) Using the following convolution integral $f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t - \tau) d\tau$.

Hint: $\sin(\alpha + \beta) = \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta)$ and $\sin(\alpha - \beta) = \sin(\alpha)\sin(\beta) - \cos(\alpha)\cos(\beta)$

Problem 5 (10 points)

You are asked to study a system using the data collected from the following experiment. The system is excited with $u(t) = \sin(\omega t)\varepsilon(t)$, and the magnitude and phase of the transfer function is recorded for different ω. Note that $\varepsilon(t)$ is the unit step function. The results are shown below.

ω	$\lvert G(j\omega)\rvert$
0.01	200
0.1	20
1	2
10	0.143
100	2.10^{-3}
1000	2.10^{-5}

Table 1: Magnitude

ω	$\angle G(j\omega)$
0.01	-90°
0.1	-91°
1	-96°
10	-135°
100	-175°
1000	-179°

Table 2: Phase

1. (8 points) Write down the transfer function of the system.
2. (2 points) How would the data change if the system had a time delay in its response?

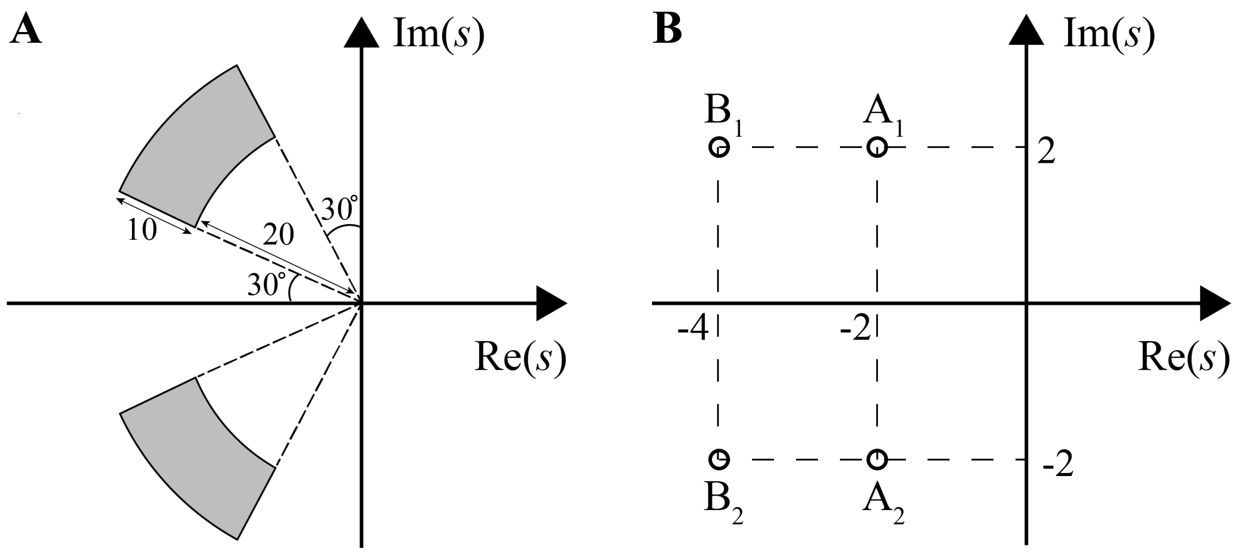
Problem 6 (25 points)

Second-order underdamped systems have a transfer function in the standard form:

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

where ζ is the damping ratio and ω_0 is the undamped natural frequency of the system. The important characteristics for the step response of the system are the rise time t_r, maximum overshoot M_p, peak time t_p, and settling time t_s that are given by:

$$t_r \simeq \frac{1.8}{\omega_0},\ M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}},\ t_p = \frac{\pi}{\overline{\omega}} \text{ and } t_s = \frac{4}{\zeta\omega_0} \text{ where } \overline{\omega} = \omega_0\sqrt{1-\zeta^2}.$$



The following questions are independent from each other.

1. (5 points) You are asked to design the system so that the poles lie within the regions shown on the left figure (Figure A). What are the admissible values of ω_0 and ζ?

2. (5 points) How would the unit step response of the system change if we move the poles from A_1, A_2 to B_1, B_2 as shown on the right figure (Figure B)? Illustrate your answer with sketches.

3. (10 points) The transfer function of a second order system has the following form:

$$G(s) = \frac{K}{s^2 + 6s + K}$$

How would the unit step response and frequency response (i.e., Bode plot) of the system change when we decrease K, first from 25 to 16, and then from 16 to 8? Illustrate your answer with sketches.

4. (5 points) Propose a mechanical or electrical system whose transfer function is given as $G(s) = 1/(2s^2 + 3)$. Denote the input and output of the system, and the values of the physical parameters. What is the value of the damping coefficient ζ?