ME-221 Final Exam —– Spring 2021

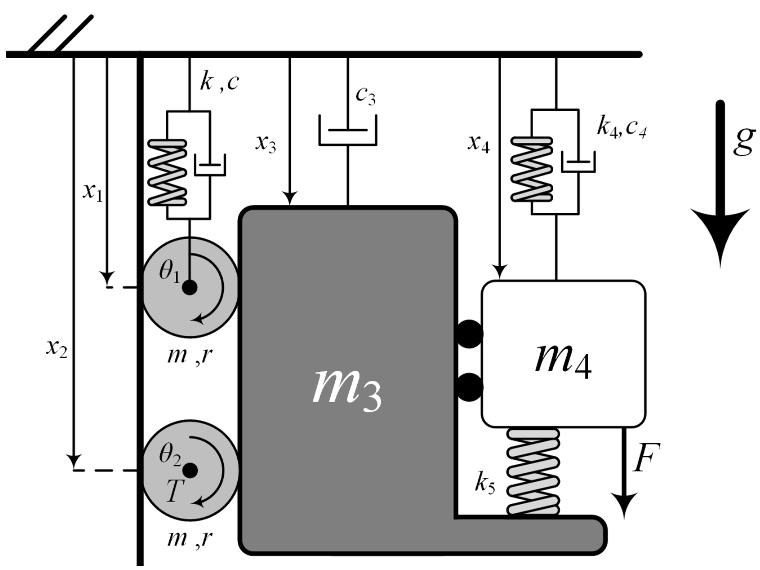
Problem 1 (25 points)

Consider the mechanical system shown below. Two cylinders (i.e., wheels) with equal mass, radius and inertia m, r and $J = mr^2/2$ roll without slipping on a rough surface. The displacement of m_3 is dictated by the movement of the cylinders because the friction coefficient between them is infinite (you can imagine them connected by a rack and pinion). Mass m_4 slips without friction on top of m_3 due to the massless black wheels. The displacement of each body is described by the coordinates x_i, $i = 1, 2, 3, 4$ in the figure. In addition, the instantaneous angle of each wheel is described by θ_1 and θ_2. The top wheel is connected to the wall through the linear spring k and linear damper c, mass m_3 is connected to the wall through a linear damper c_3, mass m_4 is connect to the wall through the linear spring k_4 and linear damper c_4, and m_3 and m_4 are connected through the linear spring k_5. The system is subjected to a gravitational field, g, and external force F is applied to m_4, and an external moment T is applied to the lower wheel. Assume that the equilibrium length of all the springs is zero.

Kinematic conditions are given as: $x_1 = r\theta_1$, $x_3 = 2r\theta_1$, and $\theta_2 = \theta_1$.

Relations between the springs, masses, and dampers are as follows:

$$k_4 = 3k,\ k_5 = 2k,\ m_3 = \frac{7}{4}m,\ m_4 = 6m,\ c_3 = \frac{1}{4}c,\ c_4 = 10c, \text{ and } r = 2.$$



1. (7 points) Write down the equations of motion using the Lagrangian method (Hint: there are only 2 equations).

2. (3 points) Find the equilibrium point of the system for $F = T = 0$.

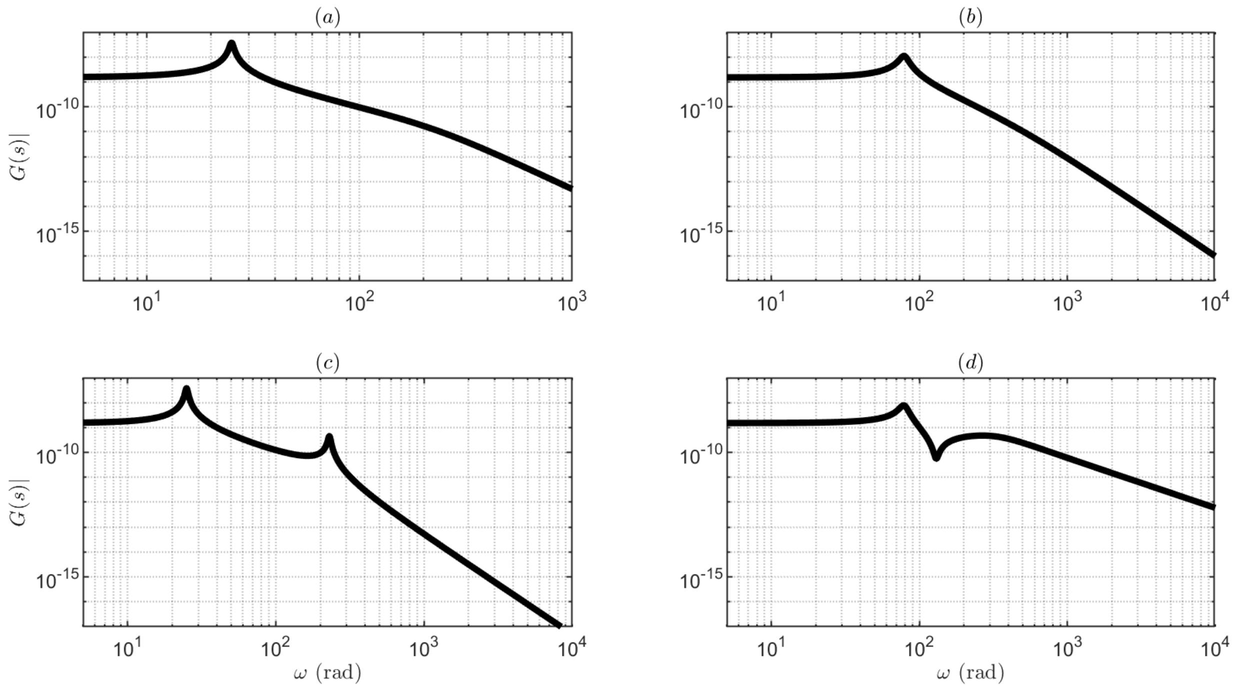
   In the following questions, the displacements are taken relative to the equilibrium point.

3. (5 points) Find the transfer functions $G(s) = X_4(s)/T(s)$.

4. (8 points) For a given set of parameters, the transfer function of the system with F as the input and x_3 as the output (i.e. $G(s) = X_3(s)/F(s)$), is calculated as:

$$G(s) = \frac{0.5}{(10s^2 + 10s + 6250)(s^2 + 368s + 52900)} \quad \text{(Hint: } \sqrt{52900} = 230\text{)}.$$

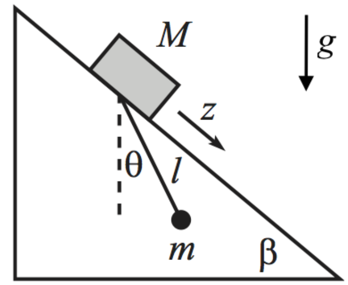
Which Bode plot shown below belongs to this transfer function? Justify your result.



5. (2 points) **Estimate** (do not calculate) the phase of dx_3/dt at the steady-state for the input $F = 2sin(7000t)$ using the information given in the previous question.

Problem 2 (10 points)

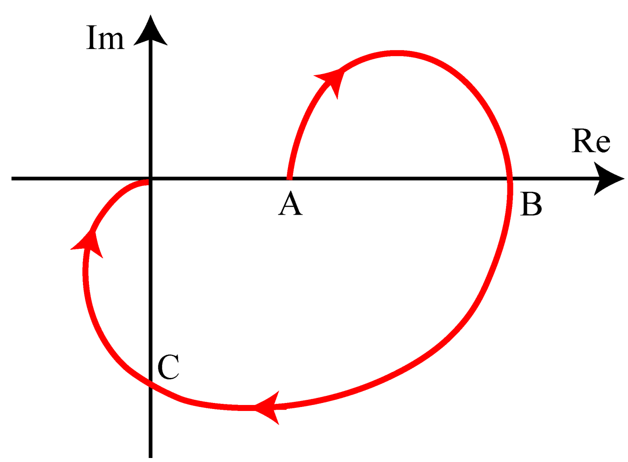
Consider the mechanical system shown on the right. A mass M is free to slide down a frictionless plane inclined at an angle β. A pendulum of length l and mass m hangs from M. Assume that M extends a short distance beyond the side of the plane so that the pendulum can hang down. Let z be the coordinate of M along the plane, and let θ be the angle of the pendulum.



1. (6 points) Derive the equations of motion (Hint: $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$).
2. (4 points) We want to linearize the motion of the pendulum for small θ. Find the equilibrium configurations of the pendulum. Note that the acceleration $\ddot{z}$ is nonzero at the equilibrium. Comment on the stability of the equilibrium states.

Problem 3 (15 points)

The Nyquist plot of an LTI system is given on the right. A, B, and C are numbers.



1. (5 points) Write down the transfer function of the system using symbols assuming that all the poles are distinct and real. What is the order of the system?
2. (10 points) If we do not have any information on the poles (real, repeated, complex etc.), explain how many different ways one could sketch the Bode plots (i.e., the slopes of the asymptotes) that complies with the given Nyquist plot. Show your work with drawings.

Problem 4 (25 points)

Solve the following three problems. They are not connected to each other.

a) (7 points) Find the inverse Laplace transform of the following function:

$$G(s) = \frac{s^4 + 2s^3 + s^2 + 2}{s^3 + 2s^2 + s}$$

b) (15 points) You are asked to study two first order systems, System I and System II, using the data collected from the following experiments. In the first experiment, System I is excited with a unit step input and the output is recorded at different time points. In the second experiment, System II is excited with $sin(\omega t)$ and the magnitude of the transfer function is recorded for different ω. The results are shown below.

t [s]	y(t)
0	0
10	3.16
20	4.33
30	4.75
40	4.91
50	4.97

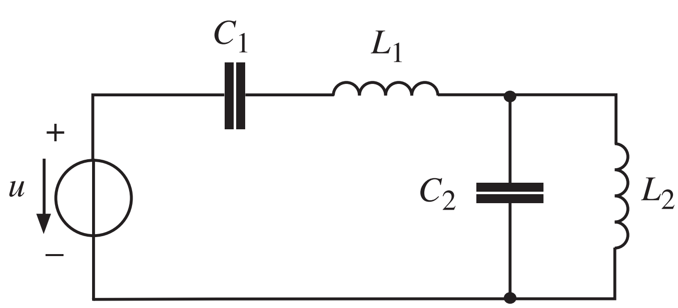
Table 1: Data from System I

ω	$\lvert G_2(j\omega)\rvert$
0.01	20
0.1	20
1	20
10	14.1
100	2
1000	0.2

Table 2: Data from System II

1. (8 points) Write down the transfer functions of the two systems, $G_1(s)$ and $G_2(s)$.
2. (2 points) Which system has a pole with a more dominant effect on the transient response?
3. (5 points) Calculate the output of System II to the input $u = e^{-10t}$.

c) (3 points) Propose a mechanical system analogous to the following electrical circuit.



Problem 5 (25 points)

Second-order underdamped systems have a transfer function in the standard form:

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

where ζ is the damping ratio and ω_0 is the undamped natural frequency of the system. The important characteristics for the step response of the system are the rise time t_r, the maximum overshoot M_p, and the setting time t_s that are given by:

$$t_r \simeq \frac{1.8}{\omega_0},\ M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \simeq 1 - \frac{\zeta}{0.6},\ \text{and}\ t_s \simeq \frac{4.6}{\zeta\omega_0}.$$

(a) (6 points) Show the regions the poles can reside to meet the following design specifications. Note that t_{rm}, M_{max}, and t_{sm} are given values.

(i) (3 points) t_r must be above t_{rm} and M_p must be lower than M_{max}.
(ii) (3 points) t_s must be lower than t_{sm}.

Now imagine that we are working with a third order system with the following transfer function:

$$H(s) = \frac{\omega_0^2}{(s^2 + 2\zeta\omega_0 s + \omega_0^2)(\tau s + 1)}$$

Remark: The transient response of a third order system is adequately represented by the second-order system approximation (i.e., H(s)=G(s)) when $|1/\tau| \geq 10|\zeta\omega_0|$.

(b) (6 points) How would the transient response of H(s) be different from G(s) for the following parameter values.

(i) (3 points) $\omega_0 = 1$, $\zeta = 0.5$ and $\tau = 0.1$.
(ii) (3 points) $\omega_0 = 1$, $\zeta = 0.5$ and $\tau = 1$.

(c) (3 points) How would the Bode plot (magnitude and phase) change if ζ is increased to 0.8?

(d) (10 points) We would like to design a first order filter with a transfer function $F(s)$ in a way that the Bode magnitude plot of the new transfer function $J(s)$, where $J(s) = H(s)F(s)$, is the one shown on the right. The parameters of $H(s)$ are given as $\omega_0 = 1$, $\zeta = 0.5$ and $\tau = 0.1$. Propose an F(s) that addresses the design specification.