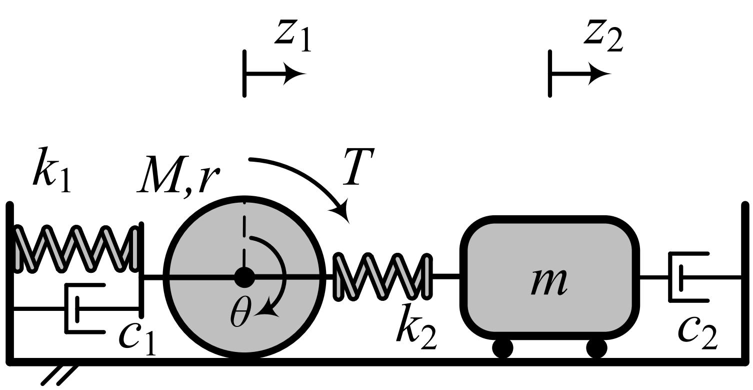
ME-221 Final Exam —— Spring 2020

Problem 1 (25 points)

Consider the mechanical system shown below. A cylinder with mass M, radius r, and inertia $J = Mr^2/2$ is rolling without slipping on a surface and the friction force is denoted by f_r (not shown in the illustration). The cylinder is connected to the wall via a spring and linear damper with spring and damping coefficients k_1 and c_1, respectively. On the other side, the cylinder is attached to a cart with mass m via a spring with spring coefficient k_2. The cart rolls on the surface without friction and it is connected to the wall via a linear damper with damping coefficient c_2. An external torque T is acting on the cylinder and the motion of the cylinder is described by the angle θ and position z_1 while the position of cart is denoted by z_2. All the coordinates are measured with respect to an equilibrium rest position (i.e. initial conditions are zero).



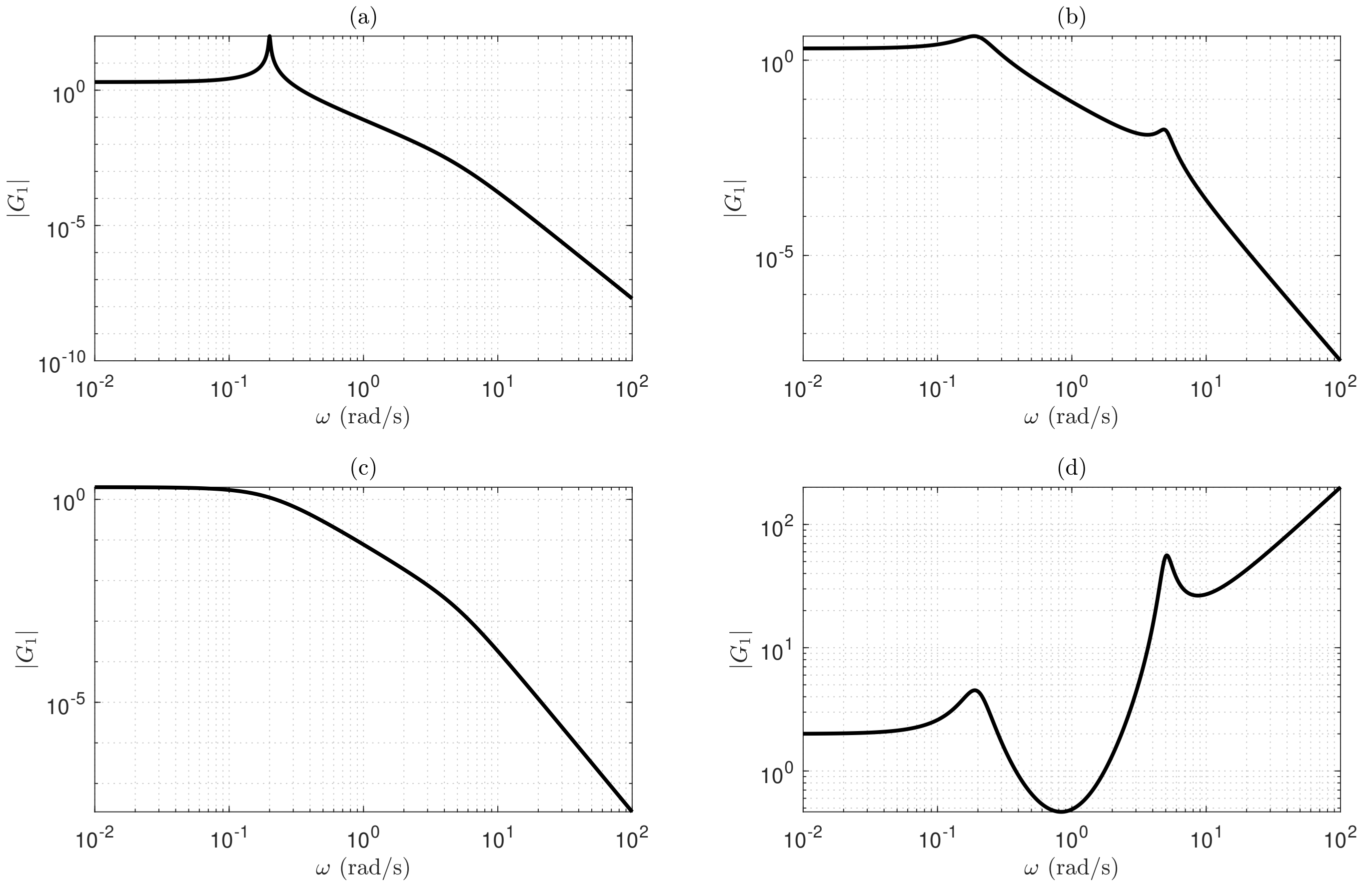
1. (6 points) Write down the equations of motion (Hint: $z_1 = r\theta$).

2. (5 points) Derive a state-space representation of the system. Assume that the input is the torque T and the output is the total force acting on the cart. What is the order of this system?

3. (6 points) Find the transfer functions $G_1(s) = Z_2(s)/T(s)$ and $G_2(s) = Z_2(s)/\Theta(s)$.

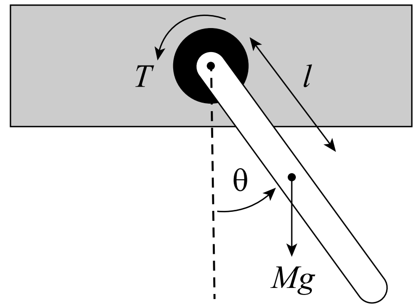
4. (8 points) For a given set of parameters, the transfer function of the system with T as input and z_2 as output (i.e. $G(s) = Z_2(s)/T(s)$), is calculated as $G(s) = \dfrac{2}{(s^2 + 0.1s + 0.04)(s^2 + s + 25)}$.

   Which Bode plot shown on the next page belongs to this system? Justify your result. **Estimate** (do not calculate) the phase shift in the steady-state value of the output for $T = 10sin(t)$.


(a)
(b)
(c)
(d)
$|G_1|$
ω (rad/s)
10^{0}
10^{-5}
10^{-10}
10^{1}
10^{2}
10^{-2}
10^{-1}

Problem 2 (20 points)

Consider the dynamics of the pendulum depicted on the right. T denotes the input torque provided by a DC motor. I is the moment of inertia of the pendulum around the pivot point and θ is the angle of rotation. The mass of the rod, M, is concentrated at its center of mass (CoM) and l is the distance from the CoM of the rod to the pivot point. The gravitational constant is denoted by g.



1. (5 points) Write down the equation of motion (Hint: $I = 4Ml^2/3$).
2. (3 points) Find the equilibrium points of the pendulum for zero input torque. Explain the physical states of the pendulum corresponding to these points.
3. (7 points) Linearize the system in the neighborhood of the stable equilibrium point using the Jacobian matrix. Derive a state-space representation of the linearized system.
4. (5 points) Suppose that we operate the DC motor in a way that it produces a desired constant torque $T = T^*$. What input torque should we apply so that we can make the pendulum at rest at any desired angle θ^*.

Problem 3 (30 points)

Solve the following three problems. They are not connected to each other.

a) You are given a linear system with input u and output y. When $u(t) = sin(t)\varepsilon(t)$ and all initial conditions are set to zero, it is found experimentally that $y(t) = 0.5[\mathrm{e}^{-t} + sin(t) - cos(t)]\varepsilon(t)$ where $\varepsilon(t)$ is the unit step function.

I. (6 points) Find the transfer function of the system.

II. (4 points) Find the output for the input $u(t) = \varepsilon(t) - \varepsilon(t-1)$ using the convolution integral.

b) Consider a linear system with the following transfer function:

$$G(s) = \frac{10(s+1)}{s(s^2+2s+100)}$$

I. (6 points) Draw the Bode plot of the system (magnitude and phase). Mark the values of the slopes and corner frequencies on the plots.

II. (4 points) Use the Bode plot to draw the Nyquist plot. Mark the approximate locations of the intersection points with the real and imaginary axes.

III. (2 points) How would the Bode (magnitude and phase) and Nyquist plots change if we had an additional e^{-s} term in the numerator of the transfer function?

c) (8 points) Consider a linear system with the following transfer function:

$$G(s) = \frac{100}{s^2+0.5s+100}$$

We would like to design a first order low-pass filter $F(s) = \dfrac{K\omega_H}{s+\omega_H}$ in such a way that the new system with the transfer function $G'(s) = G(s) \times F(s)$ has magnitude $|G'(j\omega)| = 10$ and phase angle $\phi = -2\pi/3$ at frequency $\omega = 10$ rad/sec.

I. What are the values of K and ω_H?

II. Can we ignore the additional pole $p_3 = -\omega_H$ while calculating the unit step response of the new system $G'(s)$? In other words, do you expect the unit step response of $G'(s)$ to be similar to that of the $G(s)$? Explain your reasoning.

Problem 4 (25 points)

Consider the standard second-order linear system with the following transfer function

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

where ζ is the damping ratio and ω_0 is the undamped natural frequency of the system. You are given the following performance specifications for the step response:

- The poles are in the open left-half complex plane.
- Rise time $t_r < 0.5$ seconds ($t_r = 1.8/\omega_0$).
- Percent overshoot $\%OS < 50\%$ ($\%OS = 100\mathrm{e}^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$ so $\zeta > 0.215$).

(a) (7 points) Sketch the region in the complex plane where the poles of the system must lie in order to satisfy the given specifications.

(b) (5 points) Now consider a system with the following transfer function

$$G(s) = \frac{K - 7}{s^2 + 6s + K - 7}$$

Find the values of K such that the system meets the performance specifications given in part (a).

(c) (8 points) Calculate ζ for $K = 32$. How would the step response and the Bode plot (magnitude and phase) change if we

(i) (4 points) Double the value of ζ.

(ii) (4 points) Halve the value of ζ.

Show your results by making rough sketches. Do not make calculations.

(d) (5 points) How would the settling time ($t_s = 3/\zeta\omega_0$) change if change K from 32 to 64?