ME-221
Addendum for week 5: Convolution

Linear system The output of a linear dynamical system is expressed as the superposition integral of the input and of the impulse response of the system:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t,\tau)d\tau \tag{1}$$

where $g(t,\tau)$ is the impulse response observed at time t, with an impulse applied at time τ.

Linear, time-invariant (LTI) system If the system is time-invariant, a delay in the impulse will cause the same delay in the output:

$$g(t,\tau) = g(t+\alpha, \tau+\alpha) \quad \forall\alpha \tag{2}$$

Therefore, the impulse response will not depend on t and τ independently but only on the gap between the observation time and the application of the impulse $t-\tau$.

By fixing $\alpha = -\tau$ and substituting, we get:

$$g(t,\tau) = g(t-\tau, 0) \tag{3}$$

which we will call $g(t-\tau)$.

Therefore for a time-invariant system, equation 1 can be written as:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau \tag{4}$$

LTI, initially at rest For a system that is initially at rest, inputs $u(t)$ for $t < 0$ do not influence the output. Therefore, the lower integration limit is 0:

$$y(t) = \int_{0}^{\infty} u(\tau)g(t-\tau)d\tau \tag{5}$$

LTI, initially at rest and causal In a causal system, the output only depends on past inputs. Therefore, the upper integration limit will be t, because inputs $u(\tau)$ applied at $\tau > t$ cannot influence the output. The convolution integral is therefore written as:

$$y(t) = \int_{0}^{t} u(\tau)g(t-\tau)d\tau \tag{6}$$