Math 667 final project

Please choose one problem among the following, and write up a short text (around 3-5 pages in LaTex) on it, which includes a short introduction/motivation (in the context of our class) as well as a proof of the result in question.

1. Consider the algebras $\widetilde{\mathbf{U}}^{\pm}$ and $\mathcal{V}^{\pm}$ defined on pages 13-14 of the lecture notes, for any choice of

$$\left\{\zeta_{ij}(x) \in \frac{\mathbb{K}[x^{\pm 1}]}{(1-x)^{\delta_{ij}}}\right\}_{i,j\in I}$$

(in the notation referenced, $\mathcal{V}^+ = \mathcal{V}$ and $\mathcal{V}^- = \mathcal{V}^{\mathrm{op}}$). Prove that there exists a well-defined pairing

$$\widetilde{\mathbf{U}}^+ \otimes \mathcal{V}^- \to \mathbb{K} \tag{1}$$

given by the formulas at the middle of page 17 of the lecture notes.

2. Show that the pairing (1) is the unique such pairing determined by the initial condition

$$\langle e_{i,k}, z_{j1}^{\ell}\rangle = \delta_{ij}\delta_{k+l,0}$$

and the fact that it extends to a bialgebra pairing (see page 2 of the lecture notes)

$$\widetilde{\mathbf{U}}^{\geq} \otimes \mathcal{V}^{\leq} \to \mathbb{K} \tag{2}$$

where $\left\langle \varphi_i^+(x), \varphi_j^-(y)\right\rangle = \frac{\zeta_{ji}\left(\frac{y}{x}\right)}{\zeta_{ij}\left(\frac{x}{y}\right)}$ and

$$\widetilde{\mathbf{U}}^{\geq} = \widetilde{\mathbf{U}}^{+} \bigotimes \mathbb{K}[\varphi_{i,0}^+, \varphi_{i,1}^+, \varphi_{i,2}^+, \dots]_{i\in I} \Big/ \left(\varphi_j^+(y)e_i(x) = e_i(x)\varphi_j^+(y)\frac{\zeta_{ji}\left(\frac{y}{x}\right)}{\zeta_{ij}\left(\frac{x}{y}\right)}\right)$$

$$\mathcal{V}^{\leq} = \mathcal{V}^{-} \bigotimes \mathbb{K}[\varphi_{i,0}^-, \varphi_{i,1}^-, \varphi_{i,2}^-, \dots]_{i\in I} \Big/ \left(R(\dots z_{ia}\dots)\varphi_j^-(y) = \varphi_j^-(y)R(\dots z_{ia}\dots)\prod_{i\in I, 1\leq a\leq n_i}\frac{\zeta_{ji}\left(\frac{y}{z_{ia}}\right)}{\zeta_{ij}\left(\frac{z_{ia}}{y}\right)}\right)$$

(the bialgebra structures on the tensor factors of (2) was defined on page 16 of the lecture notes).

3. Given any shuffle algebra $\mathcal{V}$ (corresponding to a choice of set I, field $\mathbb{K}$, and rational functions

$$\zeta_{ij}(x) \in \frac{\mathbb{K}[x^{\pm 1}]}{(1-x)^{\delta_{ij}}}\Big)$$

a collection

$$i_1, \dots, i_n \in I \quad \text{and} \quad \gamma_1, \dots, \gamma_n \in \mathbb{K}$$

is called a wheel if $\zeta_{i_1 i_2}(\gamma_1) = \zeta_{i_2 i_3}(\gamma_2) = \dots = \zeta_{i_n i_1}(\gamma_n) = 0$ and $\gamma_1 \dots \gamma_n = 1$. Show that

$$\mathcal{S} = \Big\{R \text{ which vanishes at } z_{i_1 a_1} = z_{i_2 a_2}\gamma_1, z_{i_2 a_2} = z_{i_3 a_3}\gamma_2, \dots, z_{i_n a_n} = z_{i_1 a_1}\gamma_n, \\ \forall a_1, \dots, a_n \geq 1 \text{ s.t. } a_x \neq a_y \text{ if } i_x = i_y\Big\}$$

is a subalgebra of $\mathcal{V}$ with respect to the shuffle product.

4. Consider representations of a doubled quiver Q with vertex set I, of some henceforth fixed dimension vector $\boldsymbol{n} = (n_i \geq 0)_{i \in I}$. If we fix for all $i \in I$ a basis

$$\mathbb{C}^{n_i} = \mathbb{C}v_{i1} \oplus \dots \oplus \mathbb{C}v_{in_i}$$

then we may regard a doubled quiver representation as an element of the quotient stack

$$S = \mathbb{A}^{\sum_{e:i\to j} 2n_i n_j} \Big/ H$$

where $H = \prod_{i \in I} \prod_{a=1}^{n_i} \mathbb{C}^*$ and the affine space above parameterizes the matrix coefficients of

$$\left\{\phi_e : \mathbb{C}^{n_i} \leftrightharpoons \mathbb{C}^{n_j} : \phi_e^*\right\}_{\forall \text{ edge } e:i\to j}$$

in our chosen basis, and the (i,a)-th copy of $\mathbb{C}^*$ in H scales the basis vector $\mathbb{C}v_{ia}$. We also have an action $T \curvearrowright S$, via dilating the linear maps ϕ_e, ϕ_e^* by certain henceforth fixed characters $t_e, \frac{q}{t_e} : T \to \mathbb{C}^*$. The gist of this construction is that

$$K_T(S) = K_{T\times H}\left(\mathbb{A}^{\sum_{e:i\to j} 2n_i n_j}\right) = \text{Rep}_T[\dots, z_{ia}^{\pm 1}, \dots]$$

and the K-theoretic Hall algebra that we've been studying lies inside the invariant part of the polynomial ring on the right with respect to the permutation of the variables.

Fix $i, j \in I$, some $a \in \{1, \dots, n_i\}$, and some $\gamma \in \text{Rep}_T$. Now consider the locally closed substack

$$U \subset \mathbb{A}^{\sum_{e:i\to j} 2n_i n_j}$$

consisting of $\{\phi_e, \phi_e^*\}_{\text{edge } e}$ for which the only non-zero matrix entries are

$$\begin{aligned}\phi_e(v_{j1}) &= x_e v_{ia}\\ \phi_e^*(v_{ia}) &= y_e v_{j2}\\ \phi_{e'}(v_{ia}) &= y_{e'} v_{j2}\\ \phi_{e'}^*(v_{j1}) &= x_{e'} v_{ia}\end{aligned}$$

where e runs over those arrows $j \to i$ such that $t_e = \gamma$ and e' runs over those arrows $i \to j$ such that $\frac{q}{t_e} = \gamma$. Moreover, in the definition of U, the x_e's and y_e's are complex numbers which are required to satisfy the relation

$$\sum_e x_e y_e - \sum_{e'} x_{e'} y_{e'} \neq 0$$

Show that the restriction map

$$K_{T\times H}\left(\mathbb{A}^{\sum_{e:i\to j} 2n_i n_j}\right) \to K_{T\times H}(U)$$

coincides with the natural quotient map

$$\text{Rep}_T[\dots, z_{ia}^{\pm 1}, \dots] \to \text{Rep}_T[\dots, z_{ia}^{\pm 1}, \dots] \Big/ \left(z_{j2} - q z_{j1}, (z_{ia} - \gamma z_{j1})^{\#}\right)$$

where $\#$ denotes the number of arrows $j \to i$ such that $t_e = \gamma$ plus the number of arrows $i \to j$ such that $\frac{q}{t_e} = \gamma$. If this all seems too complicated, first do the case $\# = 1$ and then $\# = 2$.