Koopman-based Data-driven Robust Control of Nonlinear Systems Using Integral Quadratic Constraints

Abstract— **This paper presents a novel approach for data-driven robust control of nonlinear systems using the Koopman operator. The Koopman operator theory enables the linearization of nonlinear system dynamics within a higher-dimensional space. However, the data-driven Koopman-based models are inherently approximate, influenced by various factors. To address this, our focus is on the effective characterization of modeling errors, which is essential for securing closed-loop guarantees. We utilize non-parametric Integral Quadratic Constraints (IQCs) to describe the modeling errors in a data-driven manner, modeling them as additive uncertainties in robust control design via the solution of frequency domain (FD) linear matrix inequalities (LMIs). The IQC multipliers offer a convex set representation of stabilizing robust controllers, from which we can determine the optimal robust controller. Finally, we introduce an iterative strategy that alternates between IQC multiplier identification and robust controller synthesis, ensuring a monotonic improvement in a robust performance index.**

I. INTRODUCTION

In recent years, the Koopman operator theory [1] has garnered significant attention for its ability to address the complexities of data-driven modeling and control in nonlinear systems [2], [3]. This theory offers a powerful approach by providing a global linear representation of nonlinear dynamics through the evolution of observable functions. However, achieving this global linearization often entails lifting the system to an infinite-dimensional space. To overcome this challenge, practitioners typically employ a finite-dimensional truncation of the Koopman operator, resulting in a linear but approximate representation of the system's dynamics. A popular method of obtaining linear approximations from data is Extended Dynamic Mode Decomposition (EDMD) algorithm [4].

An issue arise when Koopman theory is applied to nonlinear systems with inputs, as the linearity of lifted dynamics in observables doesn't guarantee linearity in inputs. While some approaches, like that in [5], enforce input linearity by restricting observable functions, others explore bilinear lifted models for a balance between accuracy and simplicity [6]. Despite accepting infinite-dimensional bilinear representations for input-affine systems, practical models remain approximate due to finite-dimensional truncation and data-driven methods. Thus, understanding modeling error is crucial for closed-loop guarantees.

Introduced by [7], the IQC approach offers a flexible tool for uncertain dynamical systems analysis and control. In our study, we introduce a novel approach for synthesizing robust controllers for nonlinear systems, leveraging Koopman operator theory and IQCs. Specifically, we concentrate on Linear Time-Invariant (LTI) lifted models of nonlinear systems obtained through Extended Dynamic Mode Decomposition (EDMD), relying solely on system data. To mitigate inherent modeling errors, we propose characterizing model error using IQCs. By solving FD LMIs, we identify non-parametric IQC multipliers that describe the modeling error. Subsequently, we apply the control design methodology outlined in [8] to develop controllers with robust performance assurances. Since the synthesis of robustly stabilizing controllers depends on the identified IQC multipliers, we introduce an iterative algorithm alternating between IQC multiplier identification and controller synthesis. This iterative process ensures monotonic convergence of a selected performance index.

The paper is structured as follows: Section II provides a concise overview of Koopman operator theory for autonomous system and non-autonomous systems, and present data-driven selection of observable functions and data-driven linear approximation of lifted dynamics, alongside a description of the primary problem. Section III introduce IQCs and details the proposed method. Initially, it covers the IQC-based characterization of modeling errors and the synthesis of robust controllers separately. This is followed by a discussion of the frequency sampling approach for implementing the optimization problems, and an iterative algorithm that integrates these processes. Section IV demonstrates the application of the proposed algorithm through a simulation example. The paper concludes with a brief summary in Section V.

II. PRELIMINARIES

Notations: We denote the sets of real and complex numbers as $\mathbb{R}$ and $\mathbb{C}$, respectively. the space of square summable signals of dimension p is represented by ℓ_2^p. Let $\mathcal{F}$ denote a Banach space. Identity matrix of an appropriate size is represented by I. Notations $S \succ (\succeq) 0$ and $S \prec (\preceq) 0$ signify that the matrix S is positive (-semi) definite and negative (-semi) definite, respectively. The conjugate transpose of a complex matrix S is denoted by S^* and the pseudo-inverse of S is denoted by $S^\dagger$. For a complex matrix S with full row rank, the right inverse is denoted as $S^{\mathsf{R}} = S^*(SS^*)^{-1}$. If $S \in \mathbb{C}$ is full column rank, the left inverse is represented as $S^{\mathsf{L}} = (S^*S)^{-1}S^*$. The frequency response of a discrete-time system G is indicated by $G(e^{j\omega})$.

A. Koopman operator

Firstly, consider a nonlinear autonomous system $x_{k+1} = f(x_k)$ with state $x \in \mathbb{R}^{n_x}$. The Koopman operator $\mathcal{K} : \mathcal{F} \to$

$\mathcal{F}$ associated with this system can be represented $\mathcal{K}\xi = \xi \circ f, \forall \xi \in \mathcal{F}$, i.e., $(\mathcal{K}\xi)(x_k) = \xi(f(x_k)) = \xi(x_{k+1})$. Clearly, $\mathcal{K}$ is linear operator but infinite-dimensional.

Now consider a nonlinear system with inputs,

$$H : \left\{ x_{k+1} = f(x_k, u_k), \right. \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state variable, $u \in \mathbb{R}^{n_u}$ is the input. Let $u_{k+1} = g(x_k, u_k)$ be any control law. The interconnection of the system H and the control law gives an autonomous system, which admit a Koopman operator $\mathcal{K}_{\text{cl}}$:

$$(\mathcal{K}_{\text{cl}}\xi)(x_k, u_k) := \xi(f(x_k, u_k), g(x_k, u_k)) = \xi(x_{k+1}, u_{k+1}), \tag{2}$$

In this sense, the Koopman operator $\mathcal{K}_{\text{cl}}$ globally maps the closed-loop nonlinear dynamics in the joint space of state and input to linear dynamics in the lifted space of observables. The lifted space is a Banach function space, which is generally infinite-dimensional.

B. Data-driven selection of observable functions

Suppose we have a finite data samples $\{x_k, u_k\}_{k=0}^{N-1}$, which is generated by an unknown control law $u_{k+1} = g(x_k, u_k)$. We intend to identify a finite set of observable functions $\mathcal{D} = \{\xi_j\}_{j=1}^{d}$ that is nearly linear under the Koopman operator $\mathcal{K}_{\text{cl}}$. Moreover, we hope that the finite observable functions have a special structure, i.e., $\mathcal{D} = \begin{bmatrix}\boldsymbol{\xi}(x_k) \ u_k\end{bmatrix}^\top$ with $\boldsymbol{\xi}(x_k) = \begin{bmatrix}\xi_1(x_k) \ \xi_2(x_k) \ \dots \ \xi_{d-n_u}(x_k)\end{bmatrix}^\top$, since this gives

$$\begin{bmatrix}\boldsymbol{\xi}(x_{k+1}) \\ u_{k+1}\end{bmatrix} \approx \begin{bmatrix}\mathbb{K}_{11} & \mathbb{K}_{12} \\ \mathbb{K}_{21} & \mathbb{K}_{22}\end{bmatrix} \begin{bmatrix}\boldsymbol{\xi}(x_k) \\ u_k\end{bmatrix}. \tag{3}$$

Note that $\boldsymbol{\xi}(x_{k+1}) \approx \mathbb{K}_{11}\boldsymbol{\xi}(x_k) + \mathbb{K}_{12}u_k$ is independent with the control law g, implying that this approximation is not limited to the closed-loop system between H and control law g, but applied to any system involved H. Rewrite the approaximation as

$$\boldsymbol{\xi}(x_{k+1}) = A\boldsymbol{\xi}(x_k) + Bu_k + \varepsilon_k, \tag{4}$$

where $A = \mathbb{K}_{11}$, $B = \mathbb{K}_{12}$ and ε_k denotes the one step ahead prediction error. Note that the prediction error ε_k is introduced by the restriction of the Koopman operator to a finite dimensional space as well as the structure imposed on the dictionary.

Suppose we have a set of candidate observable functions, $\boldsymbol{\xi}_c : \mathbb{R}^{n_x} \to \mathbb{R}^{n_c}$, for example,

$$\boldsymbol{\xi}_c(x) = [\underbrace{x^{(1)}, \dots, x^{(n_x)}}_{\text{mandatory}}, \underbrace{\sin x^{(i)}, x^{(i)}\cos x^{(j)}, \dots}_{\text{optional}}]^\top.$$

We hope to find $d - n_u$ observable functions from $\boldsymbol{\xi}_c$. Moreover, $\boldsymbol{\xi}$ is supposed to include the original state, since the control performance objective is often defined over the original state x_k. In this sense, we call the set of observable functions $x^{(1)}, \dots, x^{(n_x)}$ *mandatory*, and the others *optional*.

The image of $Z^+ := \begin{bmatrix}\boldsymbol{\xi}_c(x_1) \ \dots \ \boldsymbol{\xi}_c(x_{N-1})\end{bmatrix}$ is the output space of lifted dynamics. We intend to reduce the size of $\boldsymbol{\xi}_c(x)$ from n_c to $d - n_u$, which can be done by choosing the orthonormal vectors associated with $d - n_u - n_x$ largest singular values of Z^+, plus the n_x mandatory functions.

Algorithm 1: Selection of observable functions

Data: $Z^+ := \begin{bmatrix}\boldsymbol{\xi}_c(x_1) \ \dots \ \boldsymbol{\xi}_c(x_{N-1})\end{bmatrix}$
- $[U, \Sigma, V] = svd(Z^+)$
- Set $U_r = U(:, 1 : d - n_u - n_x)$
- $\boldsymbol{\xi}(x) = \begin{bmatrix} x \\ U_r^\top \boldsymbol{\xi}_c(x)\end{bmatrix}$

Result: $\boldsymbol{\xi}(x)$

C. Data-driven approximation of the Koopman operator

Based on the data samples $\{x_k, u_k\}_{k=0}^{N-1}$ and a selected dictionary of observable functions $\boldsymbol{\xi}$, EDMD [4] enables the computation of the matrices A and B in (4) by solving a least-squares problem as follows,

$$\min_{A,B} \quad \left\| Z^+ - \begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}Z \\ U\end{bmatrix}\right\| \tag{5}$$

where

$$Z := \begin{bmatrix}\boldsymbol{\xi}(x_0) \ \dots \ \boldsymbol{\xi}(x_{N-2})\end{bmatrix},$$

$$Z^+ := \begin{bmatrix}\boldsymbol{\xi}(x_1) \ \dots \ \boldsymbol{\xi}(x_{N-1})\end{bmatrix},$$

$$U := \begin{bmatrix}u_0 \ \dots \ u_{N-2}\end{bmatrix}.$$

The above problem admits a closed-loop solution $\begin{bmatrix}A & B\end{bmatrix} = Z^+ \begin{bmatrix}Z \\ U\end{bmatrix}^\dagger$.

D. Problem Formulation

Consider data $\{x_k, u_k\}_{k=0}^{N-1}$ collected from a general discrete-time nonlinear system (1) with sampling time T_s, as M trajectories of N samples. Assume the data is sufficiently informative to fully characterize the systems behaviour. Using the data and a predetermined set of observable functions $\boldsymbol{\xi}$, the discrete-time nonlinear dynamics can be approximated in the lifted space as,

$$H_0 : \left\{ \hat{\boldsymbol{\xi}}_{k+1} = A\hat{\boldsymbol{\xi}}_k + Bu_k, \right. \tag{6}$$

where $\hat{\boldsymbol{\xi}}_k \approx \boldsymbol{\xi}(x_k)$. The LTI system H_0 is an approximation of the true system H such that $H = H_0 + \Delta$, where Δ represents the error model to be treated as additive uncertainty for controller design. Thus, the interconnection of the nonlinear system H with a controller K can be represented as in Fig. 1.

Based on this, we formulate the problem of designing a data-driven controller providing closed-loop guarantees for the nonlinear system H, as the following two subproblems,

1) Characterization of the error system Δ using non-parametric dynamic IQC multipliers.
2) Synthesis of a fixed-structure controller $K = XY^{-1}$ for H_0 with guarantees of robust stability against Δ and robust performance with respect to Π_p on $w \to z$.

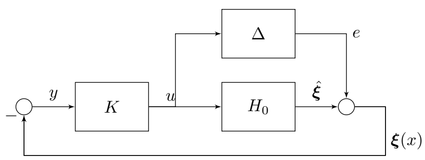

Fig. 1. Block diagram of the closed-loop system.

III. Data-driven Robust Control Design

A. Integral Quadratic Constraints

Two discrete-time signals $p(k) \in \ell_2^{n_p}[0,\infty]$ and $q(k) \in \ell_2^{n_q}[0,\infty]$ with sampling time T_s are said to satisfy the IQC defined by Π if,

$$\int_{\omega\in\Omega} \begin{bmatrix} P(e^{j\omega}) \\ Q(e^{j\omega}) \end{bmatrix}^* \underbrace{\begin{bmatrix} \Pi_{11}(e^{j\omega}) & \Pi_{12}(e^{j\omega}) \\ \Pi_{12}^*(e^{j\omega}) & \Pi_{22}(e^{j\omega}) \end{bmatrix}}_{\Pi(e^{j\omega})} \begin{bmatrix} P(e^{j\omega}) \\ Q(e^{j\omega}) \end{bmatrix} d\omega \geq 0, \tag{7}$$

where $P(e^{j\omega})$ and $Q(e^{j\omega})$ represent the discrete-time Fourier transforms of $p(k)$ and $q(k)$ respectively and $\Omega = (-\pi/T_s, \pi/T_s]$. Let Δ be a bounded causal operator. The IQC defined by Π is satisfied by Δ if, for any square summable signal p, $(p, \Delta(p))$ satisfy the IQC defined by Π.

Let $w \to z$ be a performance channel of the system H, and let $\Pi_p(\gamma)$ be a multiplier indexed by γ. The performance with respect to multiplier $\Pi_p(\gamma)$ is achieved if the IQC defined by $\Pi_p(\gamma)$ is satisfied by $H_{w\to z}$. Considering the IQC theorem [9, Corollary 3]:

Theorem 1. *The feedback interconnection of a discrete-time stable LTI system T and a bounded causal operator Δ as depicted in Fig. 2, is robustly stable against Δ and has robust performance on the channel $w \to z$ with respect to $\Pi_p(\gamma)$ if,*

1) *interconnection of T and $\tau\Delta$ is well-posed, $\forall \tau \in [0,1]$;*
2) *the IQC defined by Π is satisfied by $\tau\Delta$, $\forall \tau \in [0,1]$;*
3) *for all $\omega \in \Omega$,*

$$\begin{bmatrix} T(e^{j\omega}) \\ I \end{bmatrix}^* \Pi_{rp}(\gamma) \begin{bmatrix} T(e^{j\omega}) \\ I \end{bmatrix} \prec 0; \tag{8}$$

where,

$$\Pi_{rp}(\gamma) := \left[\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \Pi_{p,11}(\gamma) & 0 & \Pi_{p,12}(\gamma) \\ \hline \Pi_{12}^* & 0 & \Pi_{22} & 0 \\ 0 & \Pi_{p,12}^*(\gamma) & 0 & \Pi_{p,22}(\gamma) \end{array}\right]. \tag{9}$$

By [9, Remark 3] if $\Pi_{11} \succeq 0$ and $\Pi_{22} \preceq 0$, then $\tau\Delta$ satisfies the IQC defined by Π for all $\tau \in [0,1]$ if and only if Δ satisfies the IQC.

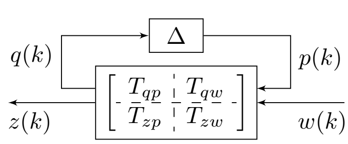

Fig. 2. General feedback interconnection.

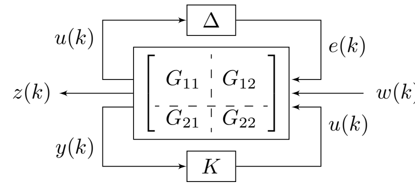

Fig. 3. Generalized plant structure of the feedback interconnection.

Fig. 2 describes a system with the interconnection of nominal closed-loop system T and error system Δ. Incorporating controller K in the nominal system T, we can transform the block diagram in Fig. 1 to a generalized plant structure as in Fig. 3 where $G_{21} = -I$ and $G_{22} = -H_0$. Then, by applying a lower linear fractional transformation to the generalized plant G and controller K, the closed-loop system can be represented as in Fig. 2 with $T = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$. Using A and B obtained by EDMD, the frequency response function (FRF) of the LTI model $H_0(e^{j\omega}) = (e^{j\omega}I - A)^{-1}B$ can be computed for any $\omega \in \Omega$. Based on $H_0(e^{j\omega})$ and following the corresponding generalized plant formulation G, the FRF $T(e^{j\omega})$ can be obtained similarly.

B. Robust Controller Synthesis

The objective of the controller synthesis is to obtain a controller structured as $K = XY^{-1}$ which guarantees robust stability against Δ and robust performance on the channel $w \to z$ with respect to $\Pi_p(\gamma)$. For some Π such that the error system Δ satisfies the IQC defined by Π, this objective can be formulated as an optimization problem,

$$\begin{aligned} \min_{K} \quad & \gamma \\ \text{s.t.} \quad & \begin{bmatrix} T \\ I \end{bmatrix}^* \Pi_{rp}(\gamma) \begin{bmatrix} T \\ I \end{bmatrix} (e^{j\omega}) \prec 0, \ \forall \omega \in \Omega, \\ & T \text{ is stable.} \end{aligned} \tag{10}$$

For $\Phi = G_{21}^{\mathsf{R}}(Y - G_{22}X)$ and $\Psi = I - \Phi\Phi^{\mathsf{L}} = G_{21}^{\mathsf{R}}G_{21}$ the closed-loop transfer function T in (10) can be written as,

$$\begin{aligned} T &= G_{11} + G_{12}X\Phi^{\mathsf{L}} = G_{11}(\Phi\Phi^{\mathsf{L}} + \Psi) + G_{12}X\Phi^{\mathsf{L}} \\ &= (G_{11}\Phi + G_{12}X)\Phi^{\mathsf{L}} + G_{11}\Psi. \end{aligned} \tag{11}$$

Since Ψ is a hermitian idempotent matrix such that,

$$\Psi\Phi = \Phi - \Phi\Phi^{\mathsf{L}}\Phi = 0 \tag{12}$$

$$\Phi^{\mathsf{L}}\Psi = \Phi^{\mathsf{L}} - \Phi^{\mathsf{L}}\Phi\Phi^{\mathsf{L}} = 0 \tag{13}$$

we get,

$$\begin{bmatrix} T \\ I \end{bmatrix} = \begin{bmatrix} G_{11}\Phi + G_{12}X & G_{11}\Psi \\ \Phi & \Psi \end{bmatrix} \begin{bmatrix} \Phi^{\mathsf{L}} \\ \Psi \end{bmatrix} = L \begin{bmatrix} \Phi^{\mathsf{L}} \\ \Psi \end{bmatrix}. \tag{14}$$

Then, by [10, Proposition 8.1.2] the first constraint in (10) can be replaced by $L^*\Pi_{rp}L \prec 0$. Using the fact that any square matrix accepts a factorisation $\Pi_{rp}(\gamma) = \Pi_{rp}^+(\gamma) + \Pi_{rp}^-(\gamma)$ with $\Pi_{rp}^+(\gamma) \succ 0$ and $\Pi_{rp}^-(\gamma) \preceq 0$, $L^*\Pi_{rp}(\gamma)L \prec 0$

can be written as $L^*\Pi_{rp}^+(\gamma)L - (-L^*\Pi_{rp}^-(\gamma)L) \prec 0$. By the Schur complement lemma, this yields the constraint,

$$\begin{bmatrix} (\Pi_{rp}^+(\gamma))^{-1} & L \\ L^* & -L^*\Pi_{rp}^-(\gamma)L \end{bmatrix} \succ 0. \tag{15}$$

The quadratic component $-L^*\Pi_{rp}^-(\gamma)L$ in (15) can be convexified around a known controller $K_c = X_c Y_c^{-1}$ such that,

$$L^*\Pi_{rp}^-(\gamma)L \preceq L^*\Pi_{rp}^-(\gamma)L_c + L_c^*\Pi_{rp}^-(\gamma)L - L_c^*\Pi_{rp}^-(\gamma)L_c \prec 0, \tag{16}$$

where

$$L_c = \begin{bmatrix} G_{11}\Phi_c + G_{12}X_c & G_{11}\Psi \\ \Phi_c & \Psi \end{bmatrix},$$
$$\Phi_c = G_{21}^{\mathsf{R}}(Y_c - G_{22}X_c).$$

By expanding, it can be seen that (16) implies,

$$\Phi^*\Pi_{rp,22}^-(\gamma)\Phi_c + \Phi_c^*\Pi_{rp,22}^-(\gamma)\Phi - \Phi_c^*\Pi_{rp,22(\gamma)}^-\Phi_c \prec 0.$$

Therefore, by [8, Lemma 1], satisfying (16) also guarantees that T is stable if $\Pi_{rp,22}^-(\gamma) \prec 0$ and K_c is nominally stabilising.

Thus, by [8, Theorem 2], for a known robustly stabilising initial controller $K_c = X_c Y_c^{-1}$ a solution of the convex problem,

$$\begin{aligned} \min_{\gamma, X, Y} \quad & \gamma \\ \text{s.t.} \quad & \begin{bmatrix} (\Pi_{rp}^+(\gamma))^{-1} & L \\ L^* & -\mathcal{L} \end{bmatrix}(e^{j\omega}) \succ 0, \ \forall \omega \in \Omega, \end{aligned} \tag{17}$$

where $\mathcal{L} = L^*\Pi_{rp}^-(\gamma)L_c + L_c^*\Pi_{rp}^-(\gamma)L - L_c^*\Pi_{rp}^-(\gamma)L_c$, is also a solution to (10) if $\Pi_{rp,22}^-(\gamma) \prec 0$, for the full proof we refer to [8]. Thus, by solving (17) for any $\Pi_{rp}^+(\gamma) \succ 0$ and $\Pi_{rp}^- \preceq 0$ such that $\Pi_{rp,22}^-(\gamma) \prec 0$, we can obtain the controller $K = XY^{-1}$ guaranteeing robust performance with index γ.

Both constraints in (10) are convexified around the initial controller K_c arriving at (17), resulting in an over approximation of a convex-concave constraint. The conservatism due to this over approximation vanishes as $K = K_c$ is attained. To achieve this, it is proposed to iteratively solve the problem in [11], replacing the initial controller at each iteration by the optimal controller obtained in the previous one which guarantees monotonic convergence of the objective to a local minimum where $K \approx K_c$ such that the conservatism vanishes.

C. Error characterization via non-parametric IQCs

In order to characterize the error system, we aim for finding a multiplier $\Pi(e^{j\omega})$ such that the input signal u and the error signal e in Fig. 1 satisfy the IQC defined by $\Pi(e^{j\omega})$ as in (7). Thus, first the frequency spectrum of the signals u and e has to be computed using the available data. To do so, we first simulate H_0 with the same input sequence used for data collection $\{u_k\}_{k=0}^{N-1}$ with initial conditions $\hat{\boldsymbol{\xi}}_0 = \boldsymbol{\xi}(x_0)$. Next, we obtain the sequence of e corresponding to the available data as $\{u_k\}_{k=0}^{N-1} = \{\boldsymbol{\xi}(x_k) - \hat{\boldsymbol{\xi}}_k\}_{k=0}^{N-1}$. Then, frequency content of e at each trajectory can be obtained as,

$$E(e^{-j\omega}) = \sum_{k=0}^{N-1} e_k e^{-j\omega T_s k}, \ \forall \omega \in \Omega. \tag{18}$$

Similarly, the frequency spectrum of the plant input u can also be computed $\forall \omega \in \Omega$ following (18).

Additionally, for a known robustly stabilising controller K_c, the IQC stability condition (8) should be satisfied by the resulting $\Pi_{rp}(\gamma)$ as in (9) where γ denotes the robust performance index achieved by K_c. Thus, for a known robustly stabilising initial controller K_c, an IQC multiplier characterizing the error system as well as minimizing the robust performance index can be obtained by solving the following FD convex optimization problem,

$$\begin{aligned} \min_{\gamma, \Pi^+, \Pi^-} \quad & \gamma \\ \text{s.t.} \quad & \int_{\omega \in \Omega} \begin{bmatrix} U \\ E \end{bmatrix}^* \Pi \begin{bmatrix} U \\ E \end{bmatrix}(e^{j\omega}) d\omega \geq 0, \\ & \begin{bmatrix} T \\ I \end{bmatrix}^* \Pi_{rp}(\gamma) \begin{bmatrix} T \\ I \end{bmatrix}(e^{j\omega}) \prec 0, \ \forall \omega \in \Omega, \\ & \Pi(e^{j\omega}) = \Pi^+(e^{j\omega}) + \Pi^-(e^{j\omega}), \ \forall \omega \in \Omega, \\ & \Pi_{11}(e^{j\omega}) \succeq 0, \ \Pi_{22}(e^{j\omega}) \preceq 0, \ \forall \omega \in \Omega, \\ & \Pi^+(e^{j\omega}) \succ 0, \ \Pi^-(e^{j\omega}) \preceq 0, \ \forall \omega \in \Omega, \\ & \Pi_{22}^-(e^{j\omega}) \prec 0, \ \forall \omega \in \Omega. \end{aligned} \tag{19}$$

Here, imposing $\Pi_{22}^- \prec 0$ in addition to $\Pi^- \preceq 0$ yields us the desired IQC multiplier such that (17) already guarantees the stability of T, with arbitrarily small conservatism added to the IQC multiplier identification problem.

Remark 1. *Due to the continuous frequency domain Ω, both problems described above are formulated as finite-dimensional convex optimization problems with an infinite number of constraints, referred to as convex semi-infinite programs (SIPs). A typical approach to solve SIPs is to sample the infinite constraints in the frequency domain at a sufficiently large set of finite frequencies $\Omega_g = \{\omega_1, \dots, \omega_g\} \subset \Omega$. Note that this sampling approach yields non-parametric IQC multipliers $\Pi(e^{j\omega})$ at a finite number of frequency points Ω_g.*

D. Iterative Approach

We propose an iterative scheme by combining the robust controller synthesis and the error characterization. The presented algorithm guarantees monotonic decrease of γ over each iteration. At the end of the algorithm a controller K that guarantees robust stability against Δ and robust performance on the channel $w \to z$ with respect to $\Pi_p(\gamma)$ is obtained by only using data trajectories collected from the system and a lifting dictionary. It should be noted that while we synthesize a linear controller in the lifted space, due to the nonlinear state transformation from the state space to the lifted space of observables the resulting controller is nonlinear in the actual state space.

Algorithm 2: Iterative algorithm over error system characterization and robust controller synthesis

Data: measured trajectories: $\{x_k, u_k\}_{k=0}^{N-1}$,
lifting functions: $\boldsymbol{\xi}(x)$,
initial robustly stabilising controller: K_c.
Preparation:
obtain A and B in (6) by EDMD.
compute $T(e^{j\omega}),\ \forall\omega \in \Omega_g$.
compute $\{(U_m, E_m)(e^{j\omega})\}_{m=1}^{M},\ \forall\omega \in \Omega_g$.
obtain RCF $K_c = X_c Y_c^{-1}$.
Iteration: set $i = 0$.
while γ *converges and* $i \leq i_{max}$ **do**
- update IQC multiplier Π:
  solve (19) for $\omega \in \Omega_g$, obtain $\Pi^+, \Pi^-\ \forall\omega \in \Omega_g$.
- update controller K:
  solve (17) for $\omega \in \Omega_g$, (iteratively as in [11]), obtain $K = XY^{-1}$.
- set $i = i + 1$.

end
Result: K, γ.

IV. Numerical Example

To illustrate the proposed method through a simulation example, we examine an inverted pendulum, a widely employed benchmark for validating nonlinear control approaches. The system dynamics are,

$$\dot{x}_1(t) = x_2(t), \tag{20}$$

$$\dot{x}_2(t) = \frac{g}{l}\sin x_1(t) - \frac{b}{ml^2}x_2(t) + \frac{1}{ml^2}u(t), \tag{21}$$

with mass $m = 1\ kg$, length $l = 1\ m$, rotational friction coefficient $b = 0.01$, and gravitational constant $g = 9.81\ m/s^2$. We discretize the dynamics using the 4^{th}-order Runge-Kutta method with sampling time $T_s = 0.01\ s$ and consider the discrete-time model as our true nonlinear system. To collect data, we simulate the discrete-time system for a single trajectory of $N = 5000$ samples with initial condition $x_0 = \begin{bmatrix}0 & 0\end{bmatrix}^T$ and a random input, such that u_k is randomly chosen from $\mathbb{U} = [-10, 10]$ with a uniform distribution for all $k \in [0, N-1]$. By also inferring some knowledge of the dynamics we choose the lifting functions $\boldsymbol{\xi}(x) = \begin{bmatrix}x_1 & x_2 & \sin(x_1)\end{bmatrix}^T$.

After applying the EDMD algorithm the lifted state matrices as in (6) are obtained, yielding a 3-dimensional stable LTI representation of the system. We consider the tracking problem where the pendulum angle x_1 is desired to track the reference w. The performance channel output is defined as $z = \begin{bmatrix}(W_1(w - x_1))^T & (W_2 u)^T\end{bmatrix}^T$ such that the tracking error as well as the control input are penalized during control design. For optimising a desired tracking response we use a low-pass filter W_1 defined by the Matlab command `W1 = 1/makeweight(0.001,1,2,Ts)` and we set $W_2 = 0.1$. We select,

$$\Pi_p = \begin{bmatrix}\gamma^2 I & 0 \\ 0 & -I\end{bmatrix}, \tag{22}$$

such that minimizing $\mathcal{H}_\infty$ norm of T_{zw} is our objective. Next, applying Algorithm 1 with initial controller $K_c = 0$, yields the state feedback controller $K = \begin{bmatrix}82.98 & 9.076 & -10.64\end{bmatrix}$ with robust performance index $\gamma^* = 9.6613$.

To observe the benefits of Koopman lifting and error characterization via non-parametric IQCs separately, we consider two other control design methods. For the same robust performance objective, first we consider the case where we did not employ lifting such that $\boldsymbol{\xi}(x) = \begin{bmatrix}x_1 & x_2\end{bmatrix}$. After identifying the system matrices by solving the EDMD problem, we use Algorithm 1 for robust controller synthesis. This approach yields a robust performance guarantee with index $\gamma_1^* = 31.5295$ achieved by the linear state feedback controller $K_1 = \begin{bmatrix}278.6 & 22.76\end{bmatrix}$. Next, to observe the benefit of using non-parametric IQCs, we follow the approach in [12] for the same performance objective. As lifting functions we again use $\boldsymbol{\xi}(x) = \begin{bmatrix}x_1 & x_2 & \sin(x_1)\end{bmatrix}^T$, which yields the same lifted representation obtained earlier. Considering the single measured trajectory, we find a lower bound on the error systems worst case ℓ_2-gain by finding the minimum value of $\gamma_e > 0$ such that

$$\sum_{k=0}^{N-1} \|e_k\|^2 \leq \gamma_e^2 \sum_{k=0}^{N-1} \|u_k\|^2,$$

is satisfied. This yields the lower bound of $\gamma_e^* = 0.0753$ achieved on the worst case ℓ_2-gain of the error system. Next, we apply the linear feedback controller synthesis method from [12, Section 3] which is based on the well known small-gain theorem. This yields a performance index of $\gamma_2^* = 14.5565$. Thus, while all three approaches yield a robust controller that can track a reference in the full operation range $x_1 \in [-\pi, \pi]$, the performance guarantee that is be achieved by the proposed method is significantly better. While we only present state feedback synthesis for simplicity, the proposed method also allows for dynamic output feedback controller synthesis to be used when full state information can not be recovered.

V. Conclusion

The proposed method offers a promising avenue for robustly controlling nonlinear systems through the integration of Koopman operator theory and IQCs. Utilizing non-parametric IQC multipliers to characterize modeling errors proves to be effective, tightening uncertainty around the lifted LTI model and reducing conservatism in control design significantly. While the iterative algorithm may not converge to the global optimum, it ensures a monotonic decrease in the performance objective. This algorithm facilitates data-driven control of nonlinear systems with closed-loop guarantees using linear control methods and solving convex problems. However, it relies on the assumption that collected data fully represents system behavior within the operational region. While this assumption can often be met with sufficiently large datasets, future research aims to quantify data quality to enhance a priori guarantees. Simulation examples demonstrate the benefits of the proposed non-parametric IQC-based error characterization, highlighting the main contribution of this work. Future endeavors will extend this approach to bilinear/LPV lifted models, promising smaller modeling errors and improved closed-loop performance.

REFERENCES

[1] B. O. Koopman, "Hamiltonian systems and transformation in hilbert space," *Proc. Natl. Acad. Sci. U. S. A.*, vol. 17, no. 5, pp. 315–318, 1931.

[2] A. Mauroy, I. Mezic, and Y. Susuki, *The Koopman Operator in Systems and Control Concepts, Methodologies, and Applications*, 2020.

[3] S. L. Brunton, M. Budišić, E. Kaiser, and J. N. Kutz, "Modern koopman theory for dynamical systems," *SIAM Review*, vol. 64, no. 2, pp. 229–340, 2022.

[4] M. O. Williams, I. G. Kevrekidis, and C. W. Rowley, "A data–driven approximation of the koopman operator: Extending dynamic mode decomposition," *Journal of Nonlinear Science*, vol. 25, no. 6, pp. 1307–1346, 2015.

[5] M. Korda and I. Mezić, "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control," *Automatica*, vol. 93, pp. 149–160, 2018.

[6] D. Bruder, X. Fu, and R. Vasudevan, "Advantages of bilinear koopman realizations for the modeling and control of systems with unknown dynamics," *IEEE Robotics and Automation Letters*, vol. 6, pp. 4369–4376, 2020.

[7] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 819–830, 1997.

[8] V. Gupta, E. Klauser, and A. Karimi, "Non-parametric IQC multipliers in data-driven robust controller synthesis," 2024. [Online]. Available: http://infoscience.epfl.ch/record/307954

[9] J. Veenman, C. W. Scherer, and H. Köroğlu, "Robust stability and performance analysis based on integral quadratic constraints," *European Journal of Control*, vol. 31, pp. 1–32, 2016.

[10] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*, 2nd ed. Princeton University Press, 2011.

[11] P. Schuchert, V. Gupta, and A. Karimi, "Data-driven fixed-structure frequency-based $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller design," *Automatica*, vol. 160, p. 111398, 2024.

[12] M. Eyuboglu, N. R. Powell, and A. Karimi, "Data-driven control synthesis using koopman operator: A robust approach," 2024. [Online]. Available: http://infoscience.epfl.ch/record/306594