Statistical Inference: Examination 2023

30 January 2023

Instructions: The time allotted for the examination is 180 minutes. You may answer in either English or French. No written material may be brought into the examination, but a simple calculator may be used if necessary. Full marks may be obtained with complete answers to four questions. The final mark will be based on the best four solutions.

First name:

Last name:

SCIPER number:

Exercise	Points	Indicative marks
1		/10 points
2		/10 points
3		/10 points
4		/10 points
5		/10 points
Total:		/40 points

Some formulae

Definition 1 *The moment-generating and cumulant-generating functions of a real-valued random variable X are*

$$M_X(t) = \mathrm{E}\left(e^{tX}\right), \quad K_X(t) = \log M_X(t), \quad t \in \mathcal{T},$$

where $\mathcal{T} = \{t \in \mathbb{R} : M_X(t) < \infty\}$.

Definition 2 *A normal (or Gaussian) random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ has probability density function*

$$f(x; \mu, \sigma^2) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \sigma^2 > 0,$$

where $\phi(u) = (2\pi)^{-1/2}e^{-u^2/2}$ for $u \in \mathbb{R}$, and we also define $\Phi(x) = \int_{-\infty}^{x} \phi(u)\,\mathrm{d}u$.

Definition 3 *A gamma random variable with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$, $X \sim \mathrm{Gamma}(\alpha, \beta)$, has probability density function*

$$f(x; \alpha, \beta) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

where $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\Gamma(\alpha) = (\alpha-1)!$ when α is a positive integer, and $\Gamma(1/2) = \sqrt{\pi}$.

Definition 4 *An exponential random variable X with rate parameter β, $X \sim \exp(\beta)$, has the gamma distribution with $\alpha = 1$.*

Definition 5 *A chi-squared random variable V with ν degrees of freedom, $V \sim \chi^2_\nu$, has the gamma distribution with $\alpha = \nu/2$ and $\beta = 1/2$, and can be expressed as $V \stackrel{\mathrm{D}}{=} Z_1^2 + \cdots + Z_\nu^2$, where $Z_1, \ldots, Z_\nu \stackrel{\mathrm{iid}}{\sim} \mathcal{N}(0, 1)$.*

Question 1

(a) What is a *pivot*? Give two examples of pivots, approximate or exact, and say how they might be used to provide a confidence set for a parameter of interest.

(b) The density of discrete data Y may be expressed in the form

$$f(y;\psi,\lambda) = m(y)\exp\{t(y)\psi + w(y)^{\mathrm{T}}\lambda - k(\psi,\lambda)\},$$

where ψ is a scalar parameter of interest and λ is a nuisance parameter. Show that the conditional density of $T = t(Y)$ given $W = w(Y)$ depends only on ψ, and say how this density may be used to construct a pivot for inference on ψ.

(c) If $X_1 \sim \mathcal{N}(\lambda, 1)$ and $X_2 \sim \mathcal{N}(\lambda\psi, 1)$ are independent, show that the linear combination $\psi X_1 - X_2$ may be used to construct a pivot, and investigate the possible forms of the resulting $(1-\alpha)$ confidence set for ψ.

Question 2

(a) Briefly explain the terms *sufficient statistic* and *minimal sufficient statistic*. When is sufficiency useful?

(b) Find a minimal sufficient statistic for (α, β) based on independent exponential observations $y_1, \ldots, y_n$ with corresponding rates $\alpha + \beta x_1, \ldots, \alpha + \beta x_n$, where $\alpha, \beta > 0$ are unknown and $x_1, \ldots, x_n$ are known constants.

(c) Independent observations $y_1, \ldots, y_n$ are available from the $U(-\theta, \theta)$ density. Find the likelihood for θ, and deduce that the largest and smallest order statistics $(y_{(1)}, y_{(n)})$ are sufficient for θ. Are they minimal sufficient?

Question 3

(a) In the context of statistical hypothesis testing, explain the terms *Type I error*, *Type II error*, *size*, *power* and *critical region*.

(b) Data y are available from a statistical model with density $m(y)\exp\{y\theta - k(\theta)\}$, where $\theta \in \Theta \subset \mathbb{R}$, and it is desired to test the null hypothesis $H_0 : \theta = \theta_0$ against the alternative $\theta = \theta_1$, where $\theta_1 < \theta_0$. Find the form of the most powerful critical region for this test, and show that it does not depend on the value of θ_1, provided $\theta_1 < \theta_0$.

(c) Find the form of the critical region in (b) when $y \sim \exp(\theta)$.

Question 4 Suppose that $Y_1, \ldots, Y_n$ are independent realisations of a random variable Y whose density function $f(y;\theta)$ satisfies suitable regularity conditions and where the vector parameter θ lies in an open subset of $\mathbb{R}^d$.

(a) State the limiting distribution of the maximum likelihood estimator $\widehat{\theta}$ as $n \to \infty$, and explain how this result can be used for inference on elements of θ.

(b) Show that the log likelihood for a random sample from the Lomax distribution, given by

$$\mathrm{P}(Y > y) = \frac{\beta^\alpha}{(\beta + y)^\alpha}, \quad y > 0, \alpha, \beta > 0,$$

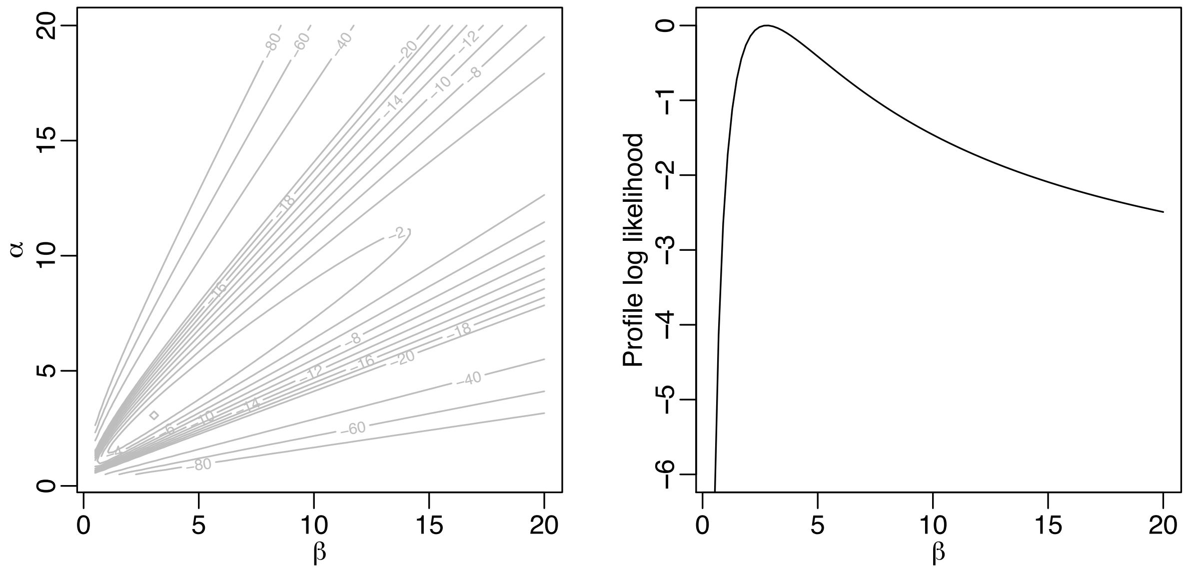
can be written in the form

$$\ell(\alpha, \beta) = n\log(\alpha/\beta) - (\alpha + 1)S(\beta), \quad \alpha, \beta > 0,$$

where $S(\beta)$ should be specified, and deduce that apart from additive constants,

$$\max_{\alpha} \log(\alpha, \beta) = -n\log S(\beta) - n\log\beta - S(\beta), \quad \beta > 0.$$

(c) The figure below shows (left panel) contours of $\ell(\alpha, \beta)$ and (right panel) $\max_\alpha \ell(\alpha, \beta)$ for $n = 100$ independent observations from the Lomax distribution. Discuss whether it would it be wise to use the result in (a) to construct confidence intervals for the parameters. If not, explain how you would obtain a $(1 - \alpha)$ confidence interval for β.



Question 5

(a) Explain what is meant by (i) proper, (ii) non-informative, and (iii) matching prior densities in the context of Bayesian inference.

(b) Independent observations $y_1, \ldots, y_n$ come from a probability density of the form

$$f(y_j \mid \theta) = m(y_j) \exp\{s(y_j)\theta - k(\theta)\}, \quad y \in \mathcal{Y}, \theta \in \Theta, .$$

and the chosen conjugate prior density is

$$\pi(\theta; a, b) = h(a, b) \exp\{a\theta - bk(\theta)\}, \quad \theta \in \Theta,$$

for known a and b. Show that the posterior density of θ given $y_1, \ldots, y_n$ is of the form $\pi(\theta; a', b')$, and give explicit forms for a' and b'.

(c) Find the posterior density for θ based on independent variables $y_1, \ldots, y_n \sim \exp(\theta)$ when $\pi(\theta) = b^a \theta^{a-1} \exp(-b\theta)/\Gamma(a)$, for $\theta > 0$ and given $a, b > 0$.

(d) Use your result from (c) to obtain $f(z \mid y_1, \ldots, y_n)$, where $z \sim \exp(x\theta)$ is conditionally independent of $y_1, \ldots, y_n$ given θ, and x is a known positive constant.

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