Since $\varepsilon > 0$ was arbitrary, this establishes (4) and (4*) and completes the proof. ∎

We mention the following uniqueness property.

$\mathscr{E} = (E_\lambda)$ is the only spectral family on $[m, M]$ that yields the representations (4) *and* (4*).

This becomes plausible if we observe that (4*) holds for every continuous real-valued function f on $[m, M]$ and the left-hand side of (4*) is defined in a way which does not depend on $\mathscr{E}$. A proof follows from a uniqueness theorem for Stieltjes integrals [cf. F. Riesz and B. Sz. -Nagy (1955), p. 111]; this theorem states that for any fixed x and y, the expression $w(\lambda) = \langle E_\lambda x, y\rangle$ is determined, up to an additive constant, by (4*) at its points of continuity and at $m-0$ and M. Since $E_M = I$, hence $\langle E_M x, y\rangle = \langle x, y\rangle$, and (E_λ) is continuous from the right, we conclude that $w(\lambda)$ is uniquely determined everywhere.

It is not difficult to see that the properties of $p(T)$ listed in Theorem 9.9-2 extend to $f(T)$; for later use we formulate this simple fact as

9.10-2 Theorem (Properties of $f(T)$). *Theorem 9.9-2 continues to hold if p, p_1, p_2 are replaced by continuous real-valued functions f, f_1, f_2 on $[m, M]$.*

9.11 Properties of the Spectral Family of a Bounded Self-Adjoint Linear Operator

It is interesting that the spectral family $\mathscr{E} = (E_\lambda)$ of a bounded self-adjoint linear operator T on a Hilbert space H reflects properties of the spectrum in a striking and simple fashion. We shall derive results of that kind from the definition of $\mathscr{E}$ (cf. Sec. 9.8) in combination with the spectral representation in Sec. 9.9.

From Sec. 9.7 we know that if H is finite dimensional, the spectral family $\mathscr{E} = (E_\lambda)$ has "points of growth" (discontinuities, jumps) precisely at the eigenvalues of T. In fact $E_{\lambda_0} - E_{\lambda_0 - 0} \neq 0$ if and only if λ_0 is an eigenvalue of T. It is remarkable, although perhaps not unexpected, that this property carries over to the infinite dimensional case:

9.11-1 Theorem (Eigenvalues). *Let $T: H \longrightarrow H$ be a bounded self-adjoint linear operator on a complex Hilbert space H and $\mathscr{E} = (E_\lambda)$ the corresponding spectral family. Then $\lambda \longmapsto E_\lambda$ has a discontinuity at any $\lambda = \lambda_0$ (that is, $E_{\lambda_0} \neq E_{\lambda_0 - 0}$) if and only if λ_0 is an eigenvalue of T. In this case, the corresponding eigenspace is*

$$\mathcal{N}(T - \lambda_0 I) = (E_{\lambda_0} - E_{\lambda_0 - 0})(H). \tag{1}$$

Proof. λ_0 is an eigenvalue of T if and only if $\mathcal{N}(T - \lambda_0 I) \neq \{0\}$, so that the first statement of the theorem follows immediately from (1). Hence it suffices to prove (1). We write simply

$$F_0 = E_{\lambda_0} - E_{\lambda_0 - 0}$$

and prove (1) by first showing that

$$F_0(H) \subset \mathcal{N}(T - \lambda_0 I) \tag{2}$$

and then

$$F_0(H) \supset \mathcal{N}(T - \lambda_0 I). \tag{3}$$

Proof of (2):

Inequality (18) in Sec. 9.8 with $\lambda = \lambda_0 - \dfrac{1}{n}$ and $\mu = \lambda_0$ is

$$\left(\lambda_0 - \frac{1}{n}\right) E(\Delta_0) \leqq TE(\Delta_0) \leqq \lambda_0 E(\Delta_0) \tag{4}$$

where $\Delta_0 = (\lambda_0 - 1/n, \lambda_0]$. We let $n \longrightarrow \infty$. Then $E(\Delta_0) \longrightarrow F_0$, so that (4) yields

$$\lambda_0 F_0 \leqq TF_0 \leqq \lambda_0 F_0.$$

Hence $TF_0 = \lambda_0 F_0$, that is, $(T - \lambda_0 I)F_0 = 0$. This proves (2).

Proof of (3):

Let $x \in \mathcal{N}(T - \lambda_0 I)$. We show that then $x \in F_0(H)$, that is, $F_0 x = x$ since F_0 is a projection.

If $\lambda_0 \notin [m, M]$, then $\lambda_0 \in \rho(T)$ by 9.2-1. Hence in this case $\mathcal{N}(T-\lambda_0 I)=\{0\} \subset F_0(H)$ since $F_0(H)$ is a vector space. Let $\lambda_0 \in [m, M]$. By assumption, $(T-\lambda_0 I)x=0$. This implies $(T-\lambda_0 I)^2 x=0$, that is, by 9.9-1,

$$\int_a^b (\lambda-\lambda_0)^2\, dw(\lambda)=0, \qquad w(\lambda)=\langle E_\lambda x, x\rangle$$

where $a<m$ and $b>M$. Here $(\lambda-\lambda_0)^2 \geqq 0$ and $\lambda \longmapsto \langle E_\lambda x, x\rangle$ is monotone increasing by 9.7-1. Hence the integral over any subinterval of positive length must be zero. In particular, for every $\varepsilon>0$ we must have

$$0=\int_a^{\lambda_0-\varepsilon} (\lambda-\lambda_0)^2\, dw(\lambda) \geqq \varepsilon^2 \int_a^{\lambda_0-\varepsilon} dw(\lambda)=\varepsilon^2 \langle E_{\lambda_0-\varepsilon}x, x\rangle$$

and

$$0=\int_{\lambda_0+\varepsilon}^b (\lambda-\lambda_0)^2\, dw(\lambda) \geqq \varepsilon^2 \int_{\lambda_0+\varepsilon}^b dw(\lambda)=\varepsilon^2\langle Ix, x\rangle-\varepsilon^2\langle E_{\lambda_0+\varepsilon}x, x\rangle.$$

Since $\varepsilon>0$, from this and 9.5-2 we obtain

$$\langle E_{\lambda_0-\varepsilon}x, x\rangle=0 \qquad \text{hence} \qquad E_{\lambda_0-\varepsilon}x=0$$

and

$$\langle x-E_{\lambda_0+\varepsilon}x, x\rangle=0 \qquad \text{hence} \qquad x-E_{\lambda_0+\varepsilon}x=0.$$

We may thus write

$$x=(E_{\lambda_0+\varepsilon}-E_{\lambda_0-\varepsilon})x.$$

If we let $\varepsilon \longmapsto 0$, we obtain $x=F_0 x$ because $\lambda \longmapsto E_\lambda$ is continuous from the right. This implies (3), as was noted before. ∎

We know that the spectrum of a bounded self-adjoint linear operator T lies on the real axis of the complex plane; cf. 9.1-3. Of course, the real axis also contains points of the resolvent set $\rho(T)$. For instance, $\lambda \in \rho(T)$ if λ is real and $\lambda<m$ or $\lambda>M$; cf. 9.2-1. It is quite remarkable that *all* real $\lambda \in \rho(T)$ can be characterized by the behavior of the spectral family in a very simple fashion. This theorem will then immediately yield a characterization of points of the continuous spectrum of T and thus complete our present discussion since the residual spectrum of T is empty, by 9.2-4.

9.11-2 Theorem (Resolvent set). *Let T and $\mathscr{E}=(E_\lambda)$ be as in Theorem 9.11-1. Then a real λ_0 belongs to the resolvent set $\rho(T)$ of T if and only if there is a $\gamma>0$ such that $\mathscr{E}=(E_\lambda)$ is constant on the interval $[\lambda_0-\gamma, \lambda_0+\gamma]$.*

Proof. In part (*a*) we prove that the given condition is sufficient for $\lambda_0 \in \rho(T)$ and in (*b*) that it is necessary. In the proof we use Theorem 9.1-2 which states that $\lambda_0 \in \rho(T)$ if and only if there exists a $\gamma>0$ such that for all $x \in H$,

$$\|(T-\lambda_0 I)x\| \geqq \gamma\|x\|. \tag{5}$$

(a) Suppose that λ_0 is real and such that $\mathscr{E}$ is constant on $J=[\lambda_0-\gamma, \lambda_0+\gamma]$ for some $\gamma>0$. By Theorem 9.9-1,

$$\|(T-\lambda_0 I)x\|^2=\langle (T-\lambda_0 I)^2 x, x\rangle=\int_{m-0}^M (\lambda-\lambda_0)^2\, d\langle E_\lambda x, x\rangle. \tag{6}$$

Since $\mathscr{E}$ is constant on J, integration over J yields the value zero, and for $\lambda \notin J$ we have $(\lambda-\lambda_0)^2 \geqq \gamma^2$, so that (6) now implies

$$\|(T-\lambda_0 I)x\|^2 \geqq \gamma^2 \int_{m-0}^M d\langle E_\lambda x, x\rangle=\gamma^2\langle x, x\rangle.$$

Taking square roots, we obtain (5). Hence $\lambda_0 \in \rho(T)$ by 9.1-2.

(b) Conversely, suppose that $\lambda_0 \in \rho(T)$. Then (5) with some $\gamma>0$ holds for all $x \in H$, so that by (6) and 9.9-1,

$$\int_{m-0}^M (\lambda-\lambda_0)^2\, d\langle E_\lambda x, x\rangle \geqq \gamma^2 \int_{m-0}^M d\langle E_\lambda x, x\rangle. \tag{7}$$

We show that we obtain a contradiction if we assume that $\mathscr{E}$ is not constant on the interval $[\lambda_0-\gamma, \lambda_0+\gamma]$. In fact, then we can find a positive $\eta<\gamma$ such that $E_{\lambda_0+\eta}-E_{\lambda_0-\eta} \neq 0$ because $E_\lambda \leqq E_\mu$ for $\lambda<\mu$ (cf. 9.7-1). Hence there is a $y \in H$ such that

$$x=(E_{\lambda_0+\eta}-E_{\lambda_0-\eta})y \neq 0.$$

We use this x in (7). Then

$$E_\lambda x = E_\lambda (E_{\lambda_0+\eta} - E_{\lambda_0-\eta})y.$$

Formula (7) in Sec. 9.7 shows that this is $(E_\lambda - E_\lambda)y = 0$ when $\lambda < \lambda_0 - \eta$ and $(E_{\lambda_0+\eta} - E_{\lambda_0-\eta})y$ when $\lambda > \lambda_0 + \eta$, hence independent of λ. We may thus take $K = [\lambda_0 - \eta, \lambda_0 + \eta]$ as the interval of integration in (7). If $\lambda \in K$, then we obtain $\langle E_\lambda x, x\rangle = \langle (E_\lambda - E_{\lambda_0-\eta})y, y\rangle$ by straightforward calculation, using again (7) in Sec. 9.7. Hence (7) gives

$$\int_{\lambda_0-\eta}^{\lambda_0+\eta} (\lambda - \lambda_0)^2 \, d\langle E_\lambda y, y\rangle \geqq \gamma^2 \int_{\lambda_0-\eta}^{\lambda_0+\eta} d\langle E_\lambda y, y\rangle.$$

But this impossible because the integral on the right is positive and $(\lambda - \lambda_0)^2 \leqq \eta^2 < \gamma^2$, where $\lambda \in K$. Hence our assumption that $\mathscr{E}$ is not constant on the interval $[\lambda_0 - \gamma, \lambda_0 + \gamma]$ is false and the proof is complete. ∎

This theorem also shows that $\lambda_0 \in \sigma(T)$ if and only if $\mathscr{E}$ is not constant in any neighborhood of λ_0 on $\mathbf{R}$. Since $\sigma_r(T) = \emptyset$ by 9.2-4 and points of $\sigma_p(T)$ correspond to discontinuities of $\mathscr{E}$ (cf. 9.11-1), we have the following theorem, which completes our discussion.

9.11-3 Theorem (Continuous spectrum). *Let T and $\mathscr{E} = (E_\lambda)$ be as in Theorem 9.11-1. Then a real λ_0 belongs to the continuous spectrum $\sigma_c(T)$ of T if and only if $\mathscr{E}$ is continuous at λ_0 (thus $E_{\lambda_0} = E_{\lambda_0-0}$) and is not constant in any neighborhood of λ_0 on $\mathbf{R}$.*

Problems

1. What can we conclude from Theorem 9.11-1 in the case of a Hermitian matrix?

2. If T in Theorem 9.11-1 is compact and has infinitely many eigenvalues, what can we conclude about (E_λ) from Theorems 9.11-1 and 9.11-2?

3. Verify that the spectral family in Prob. 7, Sec. 9.9, satisfies the three theorems in the present section.

4. We know that if m in Theorem 9.2-1 is positive then T is positive. How does this follow from the spectral representation (1), Sec. 9.9?

5. We know that the spectrum of a bounded self-adjoint linear operator is closed. How does this follow from theorems in this section?

6. Let $T: l^2 \longrightarrow l^2$ be defined by $y = (\eta_j) = Tx$ where $x = (\xi_j)$, $\eta_j = \alpha_j \xi_j$ and (α_j) is any real sequence in a finite interval $[a, b]$. Show that the corresponding spectral family (E_λ) is defined by

$$\langle E_\lambda x, y\rangle = \sum_{\alpha_j \leqq \lambda} \xi_j \bar{\eta}_j.$$

7. **(Pure point spectrum)** A bounded self-adjoint linear operator $T: H \longrightarrow H$ on a Hilbert space $H \neq \{0\}$ is said to have a *pure point spectrum* or *purely discrete spectrum* if T has an orthonormal set of eigenvectors which is total in H. Illustrate with an example that this does *not* imply $\sigma_c(T) = \emptyset$ (so that this terminology, which is generally used, may confuse the beginner for a moment).

8. Give examples of compact self-adjoint linear operators $T: l^2 \longrightarrow l^2$ having a pure point spectrum such that the set of the nonzero eigenvalues (*a*) is a finite point set, (*b*) is an infinite point set and the corresponding eigenvectors form a dense set in l^2, (*c*) is an infinite point set and the corresponding eigenvectors span a subspace of l^2 such that the orthogonal complement of the closure of that subspace is finite dimensional, (*d*) as in (*c*) but that complement is infinite dimensional. In each case find a total orthonormal set of eigenvectors.

9. **(Purely continuous spectrum)** A bounded self-adjoint linear operator $T: H \longrightarrow H$ on a Hilbert space $H \neq \{0\}$ is said to have a *purely continuous spectrum* if T has no eigenvalues. If T is any bounded self-adjoint linear operator on H, show that there is a closed subspace $Y \subset H$ which reduces T (cf. Sec. 9.6, Prob. 10) and is such that $T_1 = T|_Y$ has a pure point spectrum whereas $T_2 = T|_Z$, $Z = Y^\perp$, has a purely continuous spectrum. (This reduction facilitates the investigation of T; cf. also the remark in Sec. 9.6, Prob. 10.)

10. What can we say about the spectral families $(E_{\lambda 1})$ and $(E_{\lambda 2})$ of T_1 and T_2 in Prob. 9 in terms of the spectral family (E_λ) of T?