Homework Solution Week 9

1. (a) Pick any $u \in C_c^1(\mathbb{R})$ such that u has compact support in $(0,1)$, $\|u\| = 1$ and $\|u'\| = A > 0$. Define $u_n(x) := n^{1/2}u(nx)$, $n \in \mathbb{N}$. Then, $u_n \in C_0^1(0,1)$ because $u_n(0) = n^{1/2}u(0) = 0$, $u_n(1) = n^{1/2}u(n) = 0$ for $n \geq 1$. Moreover,

$$\|u_n\| = 1 \quad \text{and} \quad \|u_n'\| = An, \quad n \in \mathbb{N}.$$

It follows that

$$\frac{\|D_3 u_n\|}{\|u_n\|} = An \to \infty,$$

which shows that D_3 is not bounded.

(b) Since $C_0^1(0,1)$ is dense in $L^2[0,1]$, D_3^* exists. Furthermore, it follows as for D_1 in Problem 3 (a) of Week 8 that D_3 is symmetric. Now, $D_3 \subseteq D_1 \Rightarrow D_1^* \subseteq D_3^*$. Letting

$$\mathfrak{D} := \left\{ u \in AC[0,1]\,;\, u' \in L^2[0,1] \right\},$$

it follows that $\mathfrak{D} \subseteq \mathfrak{D}_{D_3^*}$. Thus, we only need to show that $\mathfrak{D}_{D_3^*} \subseteq \mathfrak{D}$ and $D_3^* v = iv'$ for all $v \in \mathfrak{D}_{D_3^*}$.

Consider $v \in \mathfrak{D}_{D_3^*}$. Since $D_3^* v \in L^2[0,1] \subset L^1[0,1]$, the function φ defined by

$$\varphi(x) := -i \int_0^x \overline{D_3^* v(y)}\, \mathrm{d}y, \quad x \in [0,1],$$

belongs to $AC[0,1]$. It follows that

$$\int_0^1 u'(\overline{-iv}) = \langle iu', v\rangle = \langle u, D_3^* v\rangle = \int_0^1 u\overline{D_3^* v} = \int_0^1 ui\varphi' = -\int_0^1 u' i\varphi = \int_0^1 u'(\overline{i\bar\varphi}), \quad \forall u \in C_0^1(0,1).$$

Hence, by the Du Bois-Reymond Lemma, there exists a constant $C \in \mathbb{C}$ such that $v = -\bar\varphi + C \in AC[0,1]$ and $D_3^* v = \overline{(i\varphi)'} = iv' \in L^2[0,1]$.

Alternative / more explicit proof : Consider $v \in \mathfrak{D}_{D_3^*}$ and an interval $[a,b] \subset [0,1]$. We will construct a sequence of functions $(u_n) \subset \mathfrak{D}_{D_3}$ such that

$$u_n'(x) \longrightarrow \delta(x-a) - \delta(x-b) \tag{1}$$

in the sense of distributions, and

$$\|u_n - \chi_{[a,b]}\| \longrightarrow 0. \tag{2}$$

It will follow that

$$\langle u_n, D_3^* v\rangle = \int_0^1 u_n \overline{D_3^* v} \longrightarrow \int_a^b \overline{D_3^* v}, \tag{3}$$

$$v \in AC[0,1] \quad \text{and} \quad \langle D_3 u_n, v\rangle = \int_0^1 iu_n' \bar v \longrightarrow -i(\bar v(b) - \bar v(a)). \tag{4}$$

Since $\langle D_3 u_n, v\rangle = \langle u_n, D_3^* v\rangle$ for all n, we will deduce that

$$\int_a^b \overline{D_3^* v} = -i(\bar v(b) - \bar v(a)), \quad \forall\, 0 < a \leqslant b < 1,$$

$v \in \mathfrak{D}$ and $D_3^* v = iv'$.

The remainder of the proof is the construction of the sequence $(u_n) \subset C_0^1(0,1)$ satisfying (1), (2), (3) and (4). Consider $\varphi \in C_0^1(-1,1)$, $\varphi > 0$, such that $\int_{-1}^1 \varphi = 1$ and let $\varphi_n(x) := n\varphi(nx)$, $n \in \mathbb{N}$. Then, for all $t \in (0,1)$, the function $\varphi_n(x-t)$ is C^1, has support in $(t - \frac{1}{n}, t + \frac{1}{n}) \subset (0,1)$ for n large enough, and satisfies

$$\int_{t-\frac{1}{n}}^{t+\frac{1}{n}} \varphi_n(x-t)\, \mathrm{d}x = 1.$$

We now define

$$u_n(x) := \int_0^x \big(\varphi_n(y-a) - \varphi_n(y-b)\big)\,\mathrm{d}y.$$

Since

$$\varphi_n(x-t) \longrightarrow \delta(x-t), \quad n \to \infty,$$

in the sense of distributions, the sequence (u_n) clearly satisfies (1). Furthermore, it is easy to see that $u_n \to \chi_{[a,b]}$ pointwise and (2) follows by dominated convergence. Finally, let us prove (4). Define, for n large enough,

$$(A_n v)(y) := \int_0^1 \varphi_n(y-x)v(x)\,\mathrm{d}x, \quad v \in L^2[0,1].$$

If $v \in C^0[0,1]$, it is not difficult to prove that $A_n v \to v$ uniformly on $[0,1]$. In particular, $A_n v \to v$ in $L^2[0,1]$. Furthermore, a simple estimate using the definition of A_n shows that

$$|\langle A_n u, v\rangle| \leqslant \|u\|\,\|v\|, \quad \forall u,v \in L^2[0,1].$$

Hence, $A_n \in \mathcal{B}(L^2[0,1])$ with $\|A_n\| \leqslant 1$. Now, given $v \in L^2[0,1]$ and $\varepsilon > 0$, consider $v_\varepsilon \in C^0[0,1]$ and $n_0 \in \mathbb{N}$ such that

$$\|v - v_\varepsilon\| < \frac{\varepsilon}{3} \quad \text{and} \quad \|A_n v_\varepsilon - v_\varepsilon\| < \frac{\varepsilon}{3}\ \forall n \geqslant n_0.$$

Then, for $n \geqslant n_0$,

$$\begin{aligned}\|A_n v - v\| &\le \|A_n(v - v_\varepsilon)\| + \|A_n v_\varepsilon - v_\varepsilon\| + \|v_\varepsilon - v\| \\ &\le \|v - v_\varepsilon\| + \|A_n v_\varepsilon - v_\varepsilon\| + \|v_\varepsilon - v\| \\ &< \varepsilon.\end{aligned}$$

This shows that $A_n v \to v$ in $L^2[0,1]$, for all $v \in L^2[0,1]$. Now, passing to a subsequence if necessary, we have that

$$\begin{aligned}\langle D_3 u_n, v\rangle &= \int_0^1 i u_n' \bar{v} = \int_0^1 i\big(\varphi_n(x-a) - \varphi_n(x-b)\big)\bar{v}(x)\,\mathrm{d}x \\ &= i\big(A_n\bar{v}(a) - A_n\bar{v}(b)\big) \\ &\longrightarrow -i\big(\bar{v}(b) - \bar{v}(a)\big),\end{aligned} \tag{5}$$

for almost every $a < b$ in $(0,1)$. It then from follows from (3) that $v \in AC[0,1]$ and $iv' = D_3^* v \in L^2[0,1]$. Hence, in fact, (5) holds for all $a < b$ in $(0,1)$, which completes the proof of (4).

2. (a) Recall the notation

$$E(f) = \int f(\lambda) dE_\lambda$$

By definition, $u \in \mathfrak{D}_{E(f)}$ if and only if

$$\int |f|^2 \,\mathrm{d}\mu_{\|E_\lambda u\|^2} < +\infty$$

Now, it suffices to prove that

$$\|E(f)u\|^2 = \int |f|^2 \,\mathrm{d}\mu_{\|E_\lambda u\|^2}$$

when $u \in \mathfrak{D}_{E(f)}$.

If f is a step function, the equality is true (see p.49 from the lecture notes). Else, we approximate f by a sequence of step functions (t_n) such that $t_n \to f$ in $L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2})$. By definition of $E(f)u$ (see again p.49), $E(f)u = \lim_{n\to+\infty} E(t_n)u$ in $\mathcal{H}$. Then

$$\int |f|^2 \,\mathrm{d}\mu_{\|E_\lambda u\|^2} = \lim_{n\to\infty} \int |t_n(\lambda)|^2 \,\mathrm{d}\mu_{\|E_\lambda u\|^2} = \lim_{n\to\infty} \|E(t_n)u\|^2 = \|E(f)u\|^2$$

(b) Suppose f bounded and let $M = \operatorname{ess\,sup}_{\mathbb{R}} |f|$. Then, for all $u \in \mathcal{H}$,

$$\int |f|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2} \leqslant M^2 \int \mathrm{d}\mu_{\|E_\lambda u\|^2} = M^2 \|u\|^2 .$$

Hence, by part (a), $u \in \mathfrak{D}_{E(f)}$ and $\|E(f)u\| \leqslant M \|u\|$. Thus, $E(f) \in \mathcal{B}(\mathcal{H})$ and $\|E(f)\| \leqslant M$.

(c) $E(1)u = \displaystyle\int \mathrm{d}E_\lambda \, u = u, \ \forall u \in \mathcal{H}$.

(d) Let $u \in \mathfrak{D}_{E(f)}$ and consider a sequence of step functions (t_j) such that $t_j \to f$ in $L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2})$. Write

$$t_j = \sum_{k=0}^{n_j} c_k^{(j)} \chi_{I_k^{(j)}}.$$

Then

$$\begin{aligned}\langle E(f)u, u\rangle &= \lim_{j\to\infty} \langle E(t_j)u, u\rangle = \lim_{j\to\infty} \sum_{k=0}^{n_j} c_k^{(j)} \left\langle E(I_k^{(j)})u, u\right\rangle = \lim_{j\to\infty} \sum_{k=0}^{n_j} c_k^{(j)} \left\| E(I_k^{(j)})u \right\|^2 \\ &= \lim_{j\to\infty} \int t_j(\lambda) \, \mathrm{d}\mu_{\|E_\lambda u\|^2} = \int f(\lambda) \, \mathrm{d}\mu_{\|E_\lambda u\|^2}.\end{aligned}$$

The last equality follows from the finiteness of the measure, $\int \mathrm{d}\mu_{\|E_\lambda u\|^2} = \|u\|^2$, which implies that $L^1(\mathbb{R}, \mu_{\|E_\lambda u\|^2})$ is continuously embedded in $L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2})$. Explicitly, by Cauchy-Schwarz,

$$\begin{aligned}\Big| \int t_j(\lambda) \, \mathrm{d}\mu_{\|E_\lambda u\|^2} - \int f(\lambda) \, \mathrm{d}\mu_{\|E_\lambda u\|^2} \Big| &\leqslant \int |t_j(\lambda) - f(\lambda)| \, \mathrm{d}\mu_{\|E_\lambda u\|^2} \\ &\leqslant \Big(\int |t_j(\lambda) - f(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2} \Big)^{1/2} \Big(\int 1 \, \mathrm{d}\mu_{\|E_\lambda u\|^2} \Big)^{1/2} \\ &= \|t_j - f\|_{L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2})} \|u\| \to 0, \quad j \to \infty.\end{aligned}$$

(e) If $u \in \mathfrak{D}_{aE(f)+bE(g)} = \mathfrak{D}_{E(f)} \cap \mathfrak{D}_{E(g)}$, then $f, g \in L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2})$ and

$$aE(f)u + bE(g)u = a \int f \, \mathrm{d}E_\lambda u + b \int g \, \mathrm{d}E_\lambda u = \int (af + bg) \, \mathrm{d}E_\lambda u.$$

Hence $af + bg \in L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2})$, i.e. $u \in \mathfrak{D}_{E(af+bg)}$. It follows that $\mathfrak{D}_{aE(f)+bE(g)} \subset \mathfrak{D}_{E(af+bg)}$ and $aE(f)u + bE(g)u = E(af + bg)u$, for all $u \in \mathfrak{D}_{aE(f)+bE(g)}$.

We have

$$f, g \in L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2}) \iff \int |f|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2} + \int |g|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2} < \infty \iff \int (|f| + |g|)^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2} < \infty,$$

hence $\mathfrak{D}_{E(f)+E(g)} = \mathfrak{D}_{E(|f|+|g|)}$.

(f) Let $u \in \mathfrak{D}_{E(f)}$ and consider a sequence of step functions (t_j) such that $t_j \to f$ in $L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2})$ as $j \to \infty$. Write

$$t_j = \sum_{k=0}^{n_j} c_k^{(j)} \chi_{I_k^{(j)}}.$$

By boundedness of E_μ we have, for all $u \in \mathfrak{D}_{E(f)}$,

$$\begin{aligned}E_\mu E(f)u &= E_\mu \lim_{j\to\infty} E(f_j)u = \lim_{j\to\infty} E_\mu E(f_j)u = \lim_{j\to\infty} E_\mu \sum_{k=0}^{n_j} c_k^{(j)} E(I_k^{(j)})u \\ &= \lim_{j\to\infty} \sum_{k=0}^{n_j} c_k^{(j)} E_\mu E(I_k^{(j)})u = \lim_{j\to\infty} \sum_{k=0}^{n_j} c_k^{(j)} E(I_k^{(j)}) E_\mu u = E(f) E_\mu u,\end{aligned}$$

hence $u \in \mathfrak{D}_{E(f)E_\mu}$. This shows that $\mathfrak{D}_{E(f)} = \mathfrak{D}_{E_\mu E(f)} \subseteq \mathfrak{D}_{E(f)E_\mu}$ and $E_\mu E(f)u = E(f)E_\mu u$ for all $u \in \mathfrak{D}_{E_\mu E(f)}$, i.e. $E_\mu E(f) \subseteq E(f)E_\mu$.

(g) **Proof of the hint :** Let $u \in \mathfrak{D}_{E(g)}$ be fixed. We first prove the hint when $f(\rho) = \sum_{k=0}^{n} c_k \chi_{I_k}(\rho)$ is a step function with $c_k \neq 0$ and the I_k are non-empty, disjoint, bounded intervals. One has

$$\int_{-\infty}^{+\infty} |f(\rho)|^2 \, \mathrm{d}\mu_{\|E_\rho E(g)u\|^2} = \sum_{k=0}^{n} |c_k|^2 \mu_{\|E_\rho E(g)u\|^2}(I_k),$$

where

$$\mu_{\|E_\rho E(g)u\|^2}(I) = \begin{cases} \|E_b E(g)u\|^2 - \|E_a E(g)u\|^2 & \text{if } I = [a,b), \\ \|E_b E(g)u\|^2 - \|E_{a+} E(g)u\|^2 & \text{if } I = (a,b), \\ \|E_{b+} E(g)u\|^2 - \|E_{a+} E(g)u\|^2 & \text{if } I = (a,b], \\ \|E_{b+} E(g)u\|^2 - \|E_a E(g)u\|^2 & \text{if } I = [a,b]. \end{cases}$$

By (f), for any $\rho \in \mathbb{R}$, we can write

$$\begin{aligned} \|E_\rho E(g)u\|^2 = \|E(g)E_\rho u\|^2 &\overset{\text{(a)}}{=} \int_{-\infty}^{+\infty} |g(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda E_\rho u\|^2} \\ &= \int_{(-\infty,\rho)} |g(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda E_\rho u\|^2} + \int_{[\rho,+\infty)} |g(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda E_\rho u\|^2} \\ &= \int_{(-\infty,\rho)} |g(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2}, \end{aligned}$$

since $E_\lambda E_\rho = E_{\min\{\lambda,\rho\}}$. A similar argument shows that

$$\|E_{\rho+} E_g u\|^2 = \int_{(-\infty,\rho]} |g(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2}$$

for any $\rho \in \mathbb{R}$. (NB: A singleton does not necessarily have zero measure.) Hence,

$$\mu_{\|E_\rho E(g)u\|^2}(I) = \int_I |g(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2}$$

for any interval I and we conclude that

$$\int_{-\infty}^{+\infty} |f(\rho)|^2 \, \mathrm{d}\mu_{\|E_\rho E(g)u\|^2} = \sum_{k=0}^{n} \int_{I_k} |c_k|^2 |g(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2} = \int_{-\infty}^{+\infty} |f(\lambda)g(\lambda)|^2 \, \mathrm{d}\mu_{\|E_\lambda u\|^2}, \ \forall u \in \mathfrak{D}_{E(g)}. \quad (6)$$

If each $c_k \geq 0$, then the equality extends to f given by a countable sum $f = \sum_{k \in \mathbb{N}} c_k \chi_{I_k}$ of disjoint intervals using Monotone Convergence. One should note that the integrals in (6) are either both finite or both infinite, but this is no issue.

Then, as any open set is the countable union of disjoint intervals, the theorem also holds for $f = \chi_U$ with U open and then for $f = \chi_S$ with S an E-measurable set using the outer regularity of the Stieltjes measures $\mu_{\|E_\rho E(g)u\|^2}$ and $\mu_{\|E_\lambda u\|^2}$. In particular, it also holds when f is a real-valued, nonnegative, simple function.

If f is a general E-measurable function (say real-valued and non-negative for simplicity, otherwise, split into real, imaginary, positive and negative parts), it suffices to take (f_n) a monotone sequence $0 \leq f_n \leq f_{n+1}$ of simple functions which converges pointwisely $\mu_{\|E_\lambda u\|^2}$-almost everywhere to f and $\mu_{\|E_\rho E(g)u\|^2}$-almost everywhere to f to conclude the proof of the hint by Monotone Convergence. For example, one can set

$$f_n(x) = \sup\{j 2^{-n} : j \in \mathbb{Z}, \ j 2^{-n} \leq \min\{f(x), 2^n\}\}$$

Proof of the equality of domains : The hint and point (a) show that if $u \in \mathfrak{D}_{E(g)}$, then u either belongs to both $\mathfrak{D}_{E(f)E(g)} \subset \mathfrak{D}_{E(g)}$ and $\mathfrak{D}_{E(fg)}$ or to neither of them, i.e.

$$\mathfrak{D}_{E(f)E(g)} = \mathfrak{D}_{E(g)} \cap \mathfrak{D}_{E(fg)}.$$

This is because either both integrals in (6) are infinite or both are finite.

Proof of $E(f)E(g) \subseteq E(fg)$ **:** An argument similar to the proof of the hint, together with point (d), shows that

$$\langle E(f)E(g)u, u\rangle = \langle E(fg)u, u\rangle\,, \quad \forall u \in \mathfrak{D}_{E(f)E(g)}. \tag{7}$$

Furthermore, if $u, v \in \mathfrak{D}_{E(f)E(g)}$, we find that

$$\langle E(f)E(g)u, v\rangle = \langle E(fg)u, v\rangle$$

as follows. Consider an E-measurable function h and let $u, v \in \mathfrak{D}_{E(h)}$. By the polarization identity (see Problem 6, Week 2), for all $\lambda \in \mathbb{R}$, we have

$$\langle E_\lambda u, v\rangle = \frac{1}{4}\big[\,\langle E_\lambda(u+v), u+v\rangle - \langle E_\lambda(u-v), u-v\rangle + i\big(\,\langle E_\lambda(u+iv), u+iv\rangle - \langle E_\lambda(u-iv), u-iv\rangle\,\big)\big].$$

It follows that $\lambda \mapsto \langle E_\lambda u, v\rangle$ is a complex-valued function of bounded variations. A corresponding (complex-valued) Lebesgue-Stieltjes measure can be associated to it. Approximating h by step functions and using the above identity, one deduces that

$$\begin{aligned}\langle E(h)u, v\rangle &= \int h(\lambda)\,\mathrm{d}\mu_{\langle E_\lambda u, v\rangle}\\ &= \frac{1}{4}\big[\,\langle E(h)(u+v), u+v\rangle - \langle E(h)(u-v), u-v\rangle\\ &\quad + i\big(\,\langle E(h)(u+iv), u+iv\rangle - \langle E(h)(u-iv), u-iv\rangle\,\big)\big],\end{aligned}$$

where the right-hand side of the first equality is the Lebesgue-Stieltjes integral of h with respect to the complex-valued measure $\mu_{\langle E_\lambda u, v\rangle}$. Hence, (7) extends to

$$\langle E(f)E(g)u, v\rangle = \langle E(fg)u, v\rangle\,, \quad \forall u, v \in \mathfrak{D}_{E(f)E(g)}.$$

Now, if f and g are bounded, then $\mathfrak{D}_{E(f)E(g)} = \mathcal{H}$ and it follows that $E(f)E(g)u = E(fg)u$. Else, it suffices to approximate $E(f)E(g)u$ and $E(fg)u$ with

$$E(f)E(g)u = \lim_{n\to\infty}\lim_{m\to\infty} E(f_n)E(g_m)u, \quad E(fg)u = \lim_{n\to\infty}\lim_{m\to\infty} E(f_n g_m)u, \tag{8}$$

where (f_n) and (g_n) are any sequences of bounded functions for which $|f_n| \le f$, $|g_n| \le g$ $\mu_{\|E_\lambda u\|^2}$-almost everywhere and $f_n \to f$, $g_n \to g$ in $L^2(\mathbb{R}, \mu_{\|E_\lambda u\|^2})$. For example, one can take

$$f_n = f\cdot\chi_{[-n,n]}\circ f = \begin{cases} f(x), & |f(x)| \le n\\ 0, & |f(x)| > n\end{cases}$$

and $g_n = g\cdot\chi_{[-n,n]}\circ g$. For each fixed n and m, as f_n and g_m are chosen bounded, one has $E(f_n)E(g_m)u = E(f_n g_m)u$ and one gets $E(f)E(g)u = E(fg)u$ at the limit.

The goal now is to prove that the double limits in (8) converges respectively to $E(f)E(g)u$ and $E(fg)u$, which is not trivial.

It follows from (a) and (e) that

$$\|(E(f_n g_m) - E(fg))\,u\|^2 \overset{\text{(e)}}{=} \|E(f_n g_m - fg)u\|^2 \overset{\text{(a)}}{=} \int_{-\infty}^{+\infty} |f_n(\rho)g_m(\rho) - f(\rho)g(\rho)|^2\,\mathrm{d}\mu_{\|E_\rho u\|^2},$$

which goes to zero as $n, m \to \infty$ by the Dominated Convergence Theorem.

Similarly, for each fixed $n \in \mathbb{N}$,

$$\lim_{m\to\infty} \|(E(f_n)E(g_m) - E(f_n)E(g))\,u\|^2 \le \lim_{m\to\infty} \|E(f_n)\|^2 \cdot \|(E(g_m) - E(g))\,u\|^2 = 0$$

and

$$\begin{aligned}\lim_{n\to\infty} \|(E(f_n)E(g) - E(f)E(g))\,u\|^2 &\overset{\text{(e)}}{=} \|E(f_n - f)E(g)u\|^2\\ &= \lim_{n\to\infty}\int_{-\infty}^{+\infty} |f_n(\rho) - f(\rho)|^2\,\mathrm{d}\mu_{\|E_\rho E(g)u\|^2}\\ &= \lim_{n\to\infty}\int_{-\infty}^{+\infty} |(f_n(\lambda) - f(\lambda))g(\lambda)|^2\,\mathrm{d}\mu_{\|E_\lambda u\|^2} = 0.\end{aligned}$$

(h) Let $u \in \mathcal{H}$ and $u_n = E(\chi_{[-n,n]} \circ f)u$, where

$$\chi_{[-n,n]} \circ f = \begin{cases} 1, & |f(x)| \le n \\ 0, & |f(x)| > n \end{cases}$$

This is well-defined because $\mathfrak{D}_{E(\chi_{[-n,n]} \circ f)} = \mathcal{H}$ by (b). Moreover, $u_n \in \mathfrak{D}_{E(f)}$ for all $n \in \mathbb{N}$ since

$$\begin{aligned}\int_{-\infty}^{+\infty} |f(\rho)|^2 \,\mathrm{d}\mu_{\left\|E_\rho E(\chi_{[-n,n]}\circ f)u\right\|^2} &= \int_{-\infty}^{+\infty} |f(\rho) \cdot \chi_{[-n,n]} \circ f(\rho)|^2 \,\mathrm{d}\mu_{\|E_\rho u\|^2} \\ &\le \int_{-\infty}^{+\infty} n^2 \,\mathrm{d}\mu_{\|E_\rho u\|^2} = n^2 \|u\|^2 < +\infty,\end{aligned}$$

where the used the hint from (g) for the first equality.

Finally, u_n converges to u because

$$\begin{aligned}\|u_n - u\|^2 = \left\|\big(E(\chi_{[-n,n]} \circ f) - E(1)\big)\, u\right\|^2 &\overset{(e)}{=} \left\|E(\chi_{[-n,n]} \circ f - 1)u\right\|^2 \\ &\overset{(a)}{=} \int_{-\infty}^{+\infty} |\chi_{[-n,n]} \circ f(\rho) - 1|^2 \,\mathrm{d}\mu_{\|E_\rho u\|^2},\end{aligned}$$

which goes to zero as $n \to \infty$ thanks to the Dominated Convergence Theorem.

(i) $\mathfrak{D}_{E(\overline{f})} = \mathfrak{D}_{E(f)}$ by (a). We prove that $E(\overline{f}) = E(f)^*$. Let $u, v \in \mathfrak{D}_{E(f)} = \mathfrak{D}_{E(\overline{f})}$. Then

$$\begin{aligned}\left\langle u, E(\overline{f})v \right\rangle = \overline{\left\langle E(\overline{f})v, u \right\rangle} = \overline{\int_{-\infty}^{+\infty} \overline{f(\lambda)} \,\mathrm{d}\mu_{\langle E_\lambda v, u\rangle}} &= \int_{-\infty}^{+\infty} f(\lambda) \,\mathrm{d}\mu_{\overline{\langle E_\lambda v, u\rangle}} \\ = \int_{-\infty}^{+\infty} f(\lambda) \,\mathrm{d}\mu_{\langle u, E_\lambda v\rangle} = \int_{-\infty}^{+\infty} f(\lambda) \,\mathrm{d}\mu_{\langle E_\lambda u, v\rangle} &= \langle E(f)u, v\rangle,\end{aligned}$$

which proves $E(\overline{f}) \subseteq E(f)^*$.

Conversely, let $v \in \mathfrak{D}_{E(f)^*}$ and consider the sequence $(v_n) \subset \mathfrak{D}_{E(\overline{f})} = \mathfrak{D}_{E(f)} \subset \mathfrak{D}_{E(f)^*}$ defined by $v_n = E(\chi_{[-n,n]} \circ f)$, as in part (h). Then

$$\begin{aligned}+\infty > \langle v, E(f)^* v\rangle = \lim_{n\to\infty} \langle v_n, E(f)^* v\rangle = \lim_{n\to\infty} \langle E(f) v_n, v\rangle &= \lim_{n\to\infty} \left\langle E(f)E(\chi_{[-n,n]} \circ f)v, v\right\rangle \\ \overset{(g)}{=} \lim_{n\to\infty} \left\langle E(f \cdot \chi_{[-n,n]} \circ f)v, v\right\rangle &= \lim_{n\to\infty} \int_{-\infty}^{+\infty} |f(\rho) \cdot \chi_{[-n,n]} \circ f(\rho)|^2 \,\mathrm{d}\mu_{\|E_\rho v\|^2} \\ = \int_{-\infty}^{+\infty} \lim_{n\to\infty} |f(\rho) \cdot \chi_{[-n,n]} \circ f(\rho)|^2 \,\mathrm{d}\mu_{\|E_\rho v\|^2} &= \int_{-\infty}^{+\infty} |f(\rho)|^2 \,\mathrm{d}\mu_{\|E_\rho v\|^2},\end{aligned}$$

where we used the Monotone Convergence Theorem to pass the limit inside the integral. This proves that $v \in \mathfrak{D}_{E(f)}$. Hence, $\mathfrak{D}_{E(f)^*} \subseteq \mathfrak{D}_{E(f)} = \mathfrak{D}_{E(\overline{f})}$.

(j) By (i), $E(f)^* = E(\overline{f})$. By (g), $\mathfrak{D}_{E(\overline{f})E(f)} = \mathfrak{D}_{E(f)} \cap \mathfrak{D}_{E(|f|^2)} = \mathfrak{D}_{E(f)E(\overline{f})}$ and $E(\overline{f})E(f) = E(|f|^2) = E(f)E(\overline{f})$ on this domain.