Homework Solution Week 8

1. One readily checks that $\mathbf{U}^* = \mathbf{U}$ and $\mathbf{V}^* = -\mathbf{V}$. That $\mathbf{U}$ and $\mathbf{V}$ are unitary then follows by observing that $I = \mathbf{U}^2$ and $I = -\mathbf{V}^2$.

2. (a) If $\overline{T}$ exists, it follows by Proposition 3.2.2 (T^* is closed) and Theorem 3.4.2 (if A is closed and densely defined, then $A^{**} = A$) that

$$T^* = \overline{T^*} = (T^*)^{**} = (T^{**})^* = \overline{T}^*.$$

(b) Suppose first that $\mathfrak{D}_{T^*}$ is dense. Then T^{**} exists and we find from Lemma 3.4.1 that

$$\mathbf{G}_{T^{**}} = \big(\mathbf{V}\mathbf{G}_{T^*}\big)^{\perp} = \big(\mathbf{V}\big(\mathbf{V}\overline{\mathbf{G}_T}\big)^{\perp}\big)^{\perp} = \big(\mathbf{V}^2\overline{\mathbf{G}_T}\big)^{\perp\perp} = -\overline{\mathbf{G}_T} = \overline{\mathbf{G}_T}.$$

This shows that T is closable and $\overline{T} = T^{**}$.

Conversely, suppose that T is closable, i.e. there exists an operator $\overline{T}$ such that $\mathbf{G}_{\overline{T}} = \overline{\mathbf{G}_T}$. Then the same argument as in the first part of the proof of Theorem 3.4.2 (replacing T by $\overline{T}$) shows that $\mathfrak{D}_{T^*}$ is dense in $\mathcal{H}$. Hence, T is closable and we deduce that $\overline{T} = T^{**}$ as above.

3. (a) For $n \geqslant 1$, consider $u_n : [0,1] \to \mathbb{R}$ defined by

$$u_n(x) = \begin{cases} nx, & x \in [0, \frac{1}{n}), \\ 2 - nx, & x \in [\frac{1}{n}, \frac{2}{n}), \\ 0, & x \in [\frac{2}{n}, 1]. \end{cases}$$

Then, $u_n \in \mathfrak{D}_{D_1}$ for all $n \geqslant 1$, and we find that

$$\|u_n\|^2 = \frac{2}{3n}, \quad \|D_1 u_n\|^2 = 2n \implies \frac{\|D_1 u_n\|}{\|u_n\|} = \sqrt{3}n, \quad n \geqslant 1.$$

Hence, D_1 is not bounded.

Since $C_0^1(0,1) \subset \mathfrak{D}_{D_1}$, and $C_0^1(0,1)$ is dense in $L^2[0,1]$, we have $\overline{\mathfrak{D}_{D_1}} = L^2[0,1]$, so D_1 is densely defined. Now, for $u, v \in \mathfrak{D}_{D_1}$, writing $\frac{\mathrm{d}}{\mathrm{d}x}(\cdot) \equiv (\cdot)'$, we have

$$\langle D_1 u, v\rangle - \langle u, D_1 v\rangle = i\int_0^1 u'\bar{v} - \int_0^1 u\overline{iv'} = i\int_0^1 u'\bar{v} + i\int_0^1 u\overline{v'} = i\int_0^1 (u\bar{v})' = iu\bar{v}\Big|_0^1 = 0. \tag{1}$$

Hence, D_1 is symmetric.

(b) Let

$$\mathfrak{D} := \left\{u \in AC[0,1]\,;\, u' \in L^2[0,1]\right\}.$$

We first observe that (1) still holds for $u \in \mathfrak{D}_{D_1}$ and $v \in \mathfrak{D}$. Hence, $\mathfrak{D} \subseteq \mathfrak{D}_{D_1^*}$ and $D_1^* v = D_1 v$ for all $v \in \mathfrak{D}$. It remains to show that $\mathfrak{D}_{D_1^*} \subseteq \mathfrak{D}$. To this aim, let $v \in \mathfrak{D}_{D_1^*}$ and consider

$$w(x) := \int_0^x D_1^* v(y)\,\mathrm{d}y + C,$$

where $C \in \mathbb{C}$ is chosen so that

$$\int_0^1 (v(x) + iw(x))\,\mathrm{d}x = 0. \tag{2}$$

(Note that the functions under the above integrals are in $L^2[0,1] \subset L^1[0,1]$.) Now, for any $u \in \mathfrak{D}_{D_1}$, an integration by parts using $w' = D_1^* v$ yields

$$\int_0^1 iu'\bar{v} = \langle D_1 u, v\rangle = \langle u, D_1^* v\rangle = \int_0^1 u\overline{D_1^* v} = uw\Big|_0^1 - \int_0^1 u'\bar{w} = i\int_0^1 iu'\bar{w} \implies \int_0^1 u'\overline{(v + iw)} = 0. \tag{3}$$

Letting

$$u(x) := \int_0^x \overline{(v(y) + iw(y))}\,\mathrm{d}y,$$

we have by (2) that

$$u \in AC[0,1], \quad u(0) = u(1) = 0, \quad u' = v + iw \in L^2[0,1].$$

Hence, $u \in \mathfrak{D}_{D_1}$ and (3) yields $\int_0^1 |v + iw|^2 = 0$, so that

$$v(x) = -iw(x) = -i\Big(\int_0^x D_1^* v(y)\,\mathrm{d}y + C\Big), \quad \text{a.e. } x \in [0,1].$$

It follows that $v \in AC[0,1]$ and $v' = -iD_1^* v \in L^2[0,1]$. Hence, $v \in \mathfrak{D}$.

(c) Since D_1 is symmetric, we know that $D_1 \subseteq \overline{D_1} = D_1^{**}$. Thus, we only need to show that $\mathfrak{D}_{D_1^{**}} \subseteq \mathfrak{D}_1$. Let $u \in \mathfrak{D}_{D_1^{**}}$. Since $D_1 \subseteq D_1^* \Rightarrow D_1^{**} \subseteq D_1^*$, we know that $u \in AC[0,1]$, $u' \in L^2[0,1]$ and $D_1^{**} u = iu'$. Hence, for any $v \in \mathfrak{D}_{D_1^*}$,

$$0 = \langle D_1^{**} u, v\rangle - \langle u, D_1^* v\rangle = \int_0^1 iu'\bar{v} - \int_0^1 u\overline{iv'} = iu\bar{v}\Big|_0^1 = i(u(1)\bar{v}(1) - u(0)\bar{v}(0)).$$

Choosing $v(x) = x$, we get $u(1) = 0$, while $v(x) = 1 - x$ yields $u(0) = 0$. Hence, $u \in \mathfrak{D}_{D_1}$.

(d) Firstly, observe that if $u, u' \in L^2(\mathbb{R})$ then

$$\begin{aligned} \big||u(x)|^2 - |u(y)|^2\big| &= \Big|\int_x^y \big(|u|^2\big)'(z)\,\mathrm{d}z\Big| = \Big|\int_x^y \big(u'\bar{u} + u\bar{u}'\big)(z)\,\mathrm{d}z\Big| \\ &\leqslant 2\int_x^y \big|u'u\big|(z)\,\mathrm{d}z \leqslant 2\Big(\int_x^y |u|^2\,\mathrm{d}z\Big)^{1/2}\Big(\int_x^y |u'|^2\,\mathrm{d}z\Big)^{1/2} \implies \\ &\lim_{x,y\to\pm\infty} |u(x)|^2 - |u(y)|^2 = 0. \end{aligned} \tag{4}$$

Thus, for any sequence $(x_n) \subset \mathbb{R}$ such that $|x_n| \to \infty$, $(|u(x_n)|^2)$ is a Cauchy sequence in $\mathbb{R}$, hence convergent. Then (4) implies that there exists a unique $\ell \geqslant 0$ such that $|u(x)|^2 \to \ell$ as $|x| \to \infty$. Since $u \in L^2(\mathbb{R})$, we must have $\ell = 0$.

Now, for any $u, v \in \mathfrak{D}_{D_2}$, we have

$$\langle D_2 u, v\rangle - \langle u, D_2 v\rangle = \int_{\mathbb{R}} iu'\bar{v} - \int_{\mathbb{R}} u\overline{iv'} = iu\bar{v}\Big|_{-\infty}^{+\infty} = 0.$$

Hence, $D_2 \subseteq D_2^*$. To prove that D_2 is selfadjoint, it remains to show that $\mathfrak{D}_{D_2^*} \subseteq \mathfrak{D}_{D_2}$. To this aim, consider $v \in \mathfrak{D}_{D_2^*}$ and a closed interval $[a,b] \subseteq \mathbb{R}$. The arguments carried out in part (b) for the interval $[0,1]$ can be extended to $[a,b]$ with obvious modifications. Therefore, the following conclusions hold: $v \in AC[a,b]$ and $v' = -iD_1^* v \in L^2[a,b]$. Since the interval $[a,b]$ is arbitrary, we indeed have $v \in \mathfrak{D}_{D_2}$.