Homework Solution Week 7

1. If T is bounded on $\mathfrak{D}_T$, consider a sequence $(u_n) \subset \mathfrak{D}_T$ such that $u_n \to u \in \overline{\mathfrak{D}_T}$. Since

$$\|Tu_n - Tu_m\| = \|T(u_n - u_m)\| \leqslant C\,\|u_n - u_m\| \to 0, \quad n, m \to \infty,$$

the sequence (Tu_n) is Cauchy in $\mathcal{H}$. Let us denote by Tu its limit. This defines a linear extension of T to $\overline{\mathfrak{D}_T}$. Defining $Tu = 0$ for $u \in \overline{\mathfrak{D}_T}^{\perp}$ and using the decomposition $\mathcal{H} = \overline{\mathfrak{D}_T} \oplus \overline{\mathfrak{D}_T}^{\perp}$ yields a bounded extension of T to $\mathcal{H}$.

2. (a) First, by (ii) and (iii), S^{-1} is defined on $\mathfrak{D}_{S^{-1}} = \mathrm{rge}(S) = \overline{\mathrm{rge}(S)} = \mathcal{H}$. Then, in view of (i), the Closed Graph Theorem yields

$$\mathbf{G}_S \text{ closed} \implies \mathbf{G}_{S^{-1}} \text{ closed} \implies S^{-1} \text{ bounded},$$

from which (iv) follows.

(b) Let $(u_n, v_n) \in \mathbf{G}_S$ such that $(u_n, v_n) \to (u, v) \in \mathcal{H}$. By (iii), $v \in \mathrm{rge}(S)$. Let $w = S^{-1}v$. By (ii) and (iii), $\mathfrak{D}_{S^{-1}} = \mathcal{H}$. Hence, (iv) implies $S^{-1} \in \mathcal{B}(\mathcal{H})$. Therefore, $w = \lim S^{-1}v_n = \lim u_n = u$, showing that $v = Su$ and $(u, v) \in \mathbf{G}_S$. Thus, $\mathbf{G}_S$ is closed.

(c) Let $v_n = Su_n \in \mathrm{rge}(S)$ such that $v_n \to v \in \mathcal{H}$. By (iv), $u_n := S^{-1}v_n$ defines a Cauchy sequence in $\mathcal{H}$. Denote by u its limit. Hence, by (i), $(u_n, v_n) \to (u, v) \in \mathbf{G}_S$, so that $v = Su \in \mathrm{rge}(S)$. Thus, $\mathrm{rge}(S)$ is closed.

Remark : Actually, for (a) and (b), (ii) it not needed. Indeed, for (a), S^{-1} is defined on the Banach Space $\mathrm{rge}(S) = \overline{\mathrm{rge}(S)}$ by (iii). Hence, the Closed Graph Theorem applies. For (b), one has $w = S^{-1}v \in \mathfrak{D}_S$, hence $||v|| = ||Sw|| \geq C||w|| = C||S^{-1}v||$ and $S^{-1} : \mathfrak{D}_{S^{-1}} \to \mathfrak{D}_S$ is bounded which is enough to carry on the proof as above.

3. Since T^* and $(T^{-1})^*$ exist, $\mathfrak{D}_T$ and $\mathfrak{D}_{T^{-1}}$ are dense in $\mathcal{H}$. Then, $\ker(T^*) = \mathrm{rge}(T)^{\perp} = \mathfrak{D}_{T^{-1}}^{\perp} = \{0\}$, so T^* is one-to-one. Consider $v \in \mathfrak{D}_{(T^*)^{-1}} = \mathrm{rge}(T^*)$. Writing $v = T^*v'$, with $v' \in \mathfrak{D}_{T^*}$, we have

$$\forall u \in \mathfrak{D}_{T^{-1}}, \quad \langle T^{-1}u, v\rangle = \langle T^{-1}u, T^*v'\rangle = \langle TT^{-1}u, v'\rangle = \langle u, (T^*)^{-1}v\rangle.$$

This shows that $v \in \mathfrak{D}_{(T^{-1})^*}$ and $(T^{-1})^*v = (T^*)^{-1}v$. Hence, $(T^*)^{-1} \subseteq (T^{-1})^*$.

Conversely, if $v \in \mathfrak{D}_{(T^{-1})^*}$ then

$$\langle T^{-1}u, v\rangle = \langle u, (T^{-1})^*v\rangle, \quad \forall u \in \mathfrak{D}_{T^{-1}}.$$

Hence, letting $u' = T^{-1}u$, we find that

$$u' \mapsto \langle Tu', (T^{-1})^*v\rangle = \langle u', v\rangle \text{ is bounded on } \mathfrak{D}_T = \mathrm{rge}(T^{-1}).$$

It follows that $(T^{-1})^*v \in \mathfrak{D}_{T^*}$ and $T^*(T^{-1})^*v = v$. In particular, $v \in \mathrm{rge}(T^*) = \mathfrak{D}_{(T^*)^{-1}}$. Hence, $\mathfrak{D}_{(T^{-1})^*} \subseteq \mathfrak{D}_{(T^*)^{-1}}$, and indeed $(T^{-1})^* = (T^*)^{-1}$.

5. (a) The (maximal) domain of definition of T is

$$\mathfrak{D}_T = \Big\{u \in L^2(\mathbb{R})\,;\, \int_{\mathbb{R}} |\phi(x)|^2 |u(x)|^2 \,\mathrm{d}x < \infty\Big\}.$$

If $\phi(x)$ is continuous, then $C_c^{\infty}(\mathbb{R}) \subset \mathfrak{D}_T$ and $\mathfrak{D}_T$ is dense.

To see that T is unbounded, let $\{E_k\}_{k\in\mathbb{N}}$ be a family of non-empty, bounded, measurable sets on which

$$|\phi(x)| \geqslant k, \quad \text{a.e. } x \in E_k,$$

and let

$$u_k = |E_k|^{-\frac{1}{2}}\chi_{E_k}(x)$$

Then

$$||u_k||_{L^2(\mathbb{R})} = |E_k|^{-\frac{1}{2}} \left(\int_{E_k} 1dx \right)^{\frac{1}{2}} = 1, \quad ||Tu_k||_{L^2(\mathbb{R})} = |E_k|^{-\frac{1}{2}} \left(\int_{E_k} |\phi(x)|^2 dx \right)^{\frac{1}{2}} \geq k$$

and we conclude that T is unbounded.

(b) We will show that $\mathfrak{D}_{T^*} = \mathfrak{D}_T$ and $T^*v = \bar{\phi}v$, for all $v \in \mathfrak{D}_{T^*}$. First, if $v \in \mathfrak{D}_T$, then $u \mapsto \langle Tu, v\rangle$ is bounded. Indeed,

$$|\langle Tu, v\rangle| = \Big| \int_{\mathbb{R}} \phi u \bar{v} \Big| \leqslant \Big(\int_{\mathbb{R}} |u|^2 \Big)^{1/2} \Big(\int_{\mathbb{R}} |\phi v|^2 \Big)^{1/2} = \|Tv\| \, \|u\| \,, \quad \forall u \in \mathfrak{D}_T.$$

Therefore, $v \in \mathfrak{D}_{T^*}$. Furthermore,

$$\langle Tu, v\rangle = \int_{\mathbb{R}} \phi u \bar{v} = \int_{\mathbb{R}} u \overline{\bar{\phi} v} = \langle u, \bar{\phi} v\rangle,$$

showing that $T^*v = \bar{\phi}v$. Hence, $\mathfrak{D}_T \subseteq \mathfrak{D}_{T^*}$ and $T^*v = \bar{\phi}v$, for all $v \in \mathfrak{D}_T$.

We now show that $\mathfrak{D}_{T^*} \subseteq \mathfrak{D}_T$. For $v \in \mathfrak{D}_{T^*}$, we have

$$\int_{\mathbb{R}} \phi u \bar{v} = \int_{\mathbb{R}} u \overline{T^* v}, \quad \forall u \in \mathfrak{D}_T.$$

Hence,

$$\langle u, \bar{\phi} v - T^* v\rangle = \int_{\mathbb{R}} u \big(\phi \bar{v} - \overline{T^* v} \big) = 0, \quad \forall u \in \mathfrak{D}_T.$$

Since $\mathfrak{D}_T$ is dense in $\mathcal{H}$, it follows that $\bar{\phi}v - T^*v = 0$, so $T^*v = \bar{\phi}v \in L^2(\mathbb{R})$. This proves that $\mathfrak{D}_{T^*} \subseteq \mathfrak{D}_T$ and we conclude that $\mathfrak{D}_{T^*} = \mathfrak{D}_T$ and $T^*v = \bar{\phi}v$, for all $v \in \mathfrak{D}_{T^*}$.

Remark: We have used the following lemma:

If $\mathfrak{D}$ is a dense subspace of $\mathcal{H}$ and $\langle u, v\rangle = 0$ for all $u \in \mathfrak{D}$, then $v = 0$.

Proof: Let $(u_n) \subset \mathfrak{D}$ such that $u_n \to v$. Then $\|v\|^2 = \langle v, v\rangle = \lim_{n\to\infty} \langle u_n, v\rangle = 0$.