Homework Solution Week 6

1. (a) The spectrum of a symmetric matrix consists of a finite number of eigenvalues. Hence, the spectral family is piecewise constant, with jump discontinuities at every eigenvalue. Furthermore, if λ_0 is an eigenvalue, we have

$$\ker(S-\lambda_0)=\operatorname{rge}(E_{\lambda_0^+}-E_{\lambda_0}).$$

We write $P_{\lambda_0}=E_{\lambda_0^+}-E_{\lambda_0}$. Let $u\in\mathcal{H}$ and define $\phi(\lambda):=\langle E_\lambda u,u\rangle$. We first observe that the distributional derivative of ϕ at $\lambda=\lambda_0$ is given by

$$\phi'(\lambda_0)=(\phi(\lambda_0^+)-\phi(\lambda_0))\delta_{\lambda_0},$$

where δ_{λ_0} denotes the Dirac mass at $\lambda=\lambda_0$. Indeed, for any $\epsilon>0$ and any $f\in C_0^\infty((\lambda_0-\epsilon,\lambda_0+\epsilon))$, one has

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\phi'(\lambda)f(\lambda)\,\mathrm{d}\lambda&=-\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\phi(\lambda)f'(\lambda)\,\mathrm{d}\lambda\\&=-\phi(\lambda_0)\int_{\lambda_0-\epsilon}^{\lambda_0}f'(\lambda)\,\mathrm{d}\lambda+\phi(\lambda_0^+)\int_{\lambda_0}^{\lambda_0+\epsilon}f'(\lambda)\,\mathrm{d}\lambda\\&=(\phi(\lambda_0^+)-\phi(\lambda_0))f(\lambda_0).\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\lambda\,\mathrm{d}\langle E_\lambda u,u\rangle&=\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\lambda\,\mathrm{d}\phi(\lambda)=(\phi(\lambda_0^+)-\phi(\lambda_0))\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\lambda\delta_{\lambda_0}(\lambda)\,\mathrm{d}\lambda\\&=\lambda_0(\phi(\lambda_0^+)-\phi(\lambda_0))=\lambda_0\left\langle(E_{\lambda_0^+}-E_{\lambda_0})u,u\right\rangle.\end{aligned}$$

We deduce that

$$S=\int_m^{M+\varepsilon}\lambda\,\mathrm{d}E_\lambda=\sum_{\lambda\in\sigma_p(S)}\lambda\,P_\lambda.$$

(b) Suppose now S is a symmetric and compact (in particular, it is bounded) operator with an infinite sequence of eigenvalues $(\lambda_n)_{n=1}^\infty\subset\mathbb{R}$. In particular, we work in the setting of an infinitely-dimension Hilbert space since there are infinitely many eigenvalues. Each eigenspace E_λ corresponding to a nonzero eigenvalue λ is finite-dimensional because the restriction $S_{|E_\lambda}$ of S to that eigenspace must be compact and is given by a multiple of the identity, $S_{|E_\lambda}(x)=\lambda x$. However, the identity is compact only in a finite-dimensional space. Furthermore, 0 must be in the spectrum. Otherwise, S would have a bounded inverse S^{-1} and it is easy to check that the product $S^{-1}S=I$ of a bounded operator with a compact operator must be compact. However, the identity I is not compact in a vector space of infinite dimension, which leads to a contradiction.

Hence,

$$0\in\sigma(S),\quad \sigma(S)\setminus\{0\}=\sigma_p(S)\setminus\{0\},\quad 0\in\overline{(\lambda_n)_{n=1}^\infty}.$$

Note that both $0\in\sigma_p(S)$ and $0\notin\sigma_p(S)$ can occur. In any case, given any $\epsilon>0$ small enough, the spectral decomposition of S splits into

$$S=\int_{-\epsilon}^{\epsilon}\lambda\,\mathrm{d}E_\lambda+\sum_{\lambda\in\sigma_p(S),\,|\lambda|>\epsilon}\lambda\,P_\lambda.$$

Here, the second term is a finite sum, while the first satisfies

$$\Big\|\int_{-\epsilon}^{\epsilon}\lambda\,\mathrm{d}E_\lambda\Big\|\leqslant 2\epsilon^2.$$

Hence,

$$\Big\| S - \sum_{\lambda \in \sigma_p(S),\, |\lambda| > \epsilon} \lambda\, P_\lambda \Big\| \leqslant 2\epsilon^2,$$

showing that the series $\displaystyle\sum_{\lambda \in \sigma_p(S),\, \lambda \neq 0} \lambda\, P_\lambda$ converges to S.

Choosing an orthonormal basis in each subspace $\mathrm{rge}(P_\lambda)$ ($\lambda \neq 0$ in case (b)), one recovers the spectral theorems for symmetric matrices and for symmetric compact operators, respectively.

3/4. See the proof in Kreyszig's book. One should note that the book uses a different convention. They choose the spectral family to be right-continuous. This is not an issue. From a right-continuous family F_t, one gets a left-continuous family

$$E_\lambda = \lim_{t \to \lambda^-} F_t$$

and conversely, from a left-continuous family E_λ, one gets a right-continuous family

$$F_t = \lim_{\lambda \to t^+} E_\lambda$$

The families have the same points of continuity and the same jumps.

5. (a) Let $f \in C^0([m, M])$ where m, M are the lower/upper bounds of S. It is shown in Lemma 2.1.2 that

$$A_f = \int_m^{M^+} f(\lambda) dE_\lambda = \lim_{n \to +\infty} \sum_{k=1}^{n} f\left(m + k \cdot \frac{M + \varepsilon - m}{n}\right) \left(E_{\left(m + k \cdot \frac{M+\varepsilon-m}{n}\right)} - E_{\left(m + (k-1) \cdot \frac{M+\varepsilon-m}{n}\right)}\right)$$

for any $\varepsilon > 0$ with the partition

$$m = \lambda_0 < \lambda_1 < ... < \lambda_n = M + \varepsilon, \quad \lambda_k = m + k \cdot \frac{M + \varepsilon - m}{n}$$

is a well-defined operator.

For $u \in \mathcal{H}$, $||u|| = 1$, one has

$$\begin{aligned}
\langle A_f u, u \rangle &= \lim_{n \to +\infty} \sum_{k=1}^{n} f\left(m + k \cdot \frac{M + \varepsilon - m}{n}\right) \underbrace{\left\langle \left(E_{\left(m + k \cdot \frac{M+\varepsilon-m}{n}\right)} - E_{\left(m + (k-1) \cdot \frac{M+\varepsilon-m}{n}\right)}\right) u, u \right\rangle}_{\geq 0} \\
&\leq \lim_{n \to +\infty} \sum_{k=1}^{n} \max_{x \in [m, M+\varepsilon]} |f(x)| \left\langle \left(E_{\left(m + k \cdot \frac{M+\varepsilon-m}{n}\right)} - E_{\left(m + (k-1) \cdot \frac{M+\varepsilon-m}{n}\right)}\right) u, u \right\rangle \\
&= \max_{x \in [m, M+\varepsilon]} |f(x)| \langle u, u \rangle = \max_{x \in [m, M+\varepsilon]} |f(x)|
\end{aligned}$$

and similarly

$$\langle A_f u, u \rangle \geq - \max_{x \in [m, M+\varepsilon]} |f(x)|$$

Letting $\varepsilon \to 0^+$ shows that

$$||A_f|| = \sup_{||u||=1} | \langle A_f u, u \rangle | \leq \max_{x \in [m, M]} |f(x)|$$

If $p(x) = a_0 + a_1 x + ... + a_n x^n$ is a real polynomial, it is shown in Lemma 2.2.2 that

$$p(S) = a_0 + a_1 S + ... + a_n S^n = A_p = \int_m^{M^+} p(\lambda) dE_\lambda$$

If $p_n \to f$ uniformly in $C^0([m, M])$-norm, then p_n is Cauchy in $C^0([m, M])$, i.e. for $\varepsilon > 0$, there is $N \in \mathbb{N}$ for which

$$n_1, n_2 \geq N \Longrightarrow \max_{x \in [m, M]} |p_{n_1}(x) - p_{n_2}(x)| \leq \varepsilon$$

It follows directly that

$$n_1, n_2 \geq N \Longrightarrow ||p_{n_1}(S) - p_{n_2}(S)|| \leq \max_{x\in[m,M]} |p_{n_1}(x) - p_{n_2}(x)| \leq \varepsilon$$

Hence $p_n(S)$ is Cauchy and converges to an operator which we call $f(S)$.

(b) The goal is to show that the limit $f(S)$ is

$$A_f = \int_m^{M^+} f(\lambda) dE_\lambda$$

from point (a). One has

$$A_f = \int_m^{M^+} f(\lambda) dE_\lambda = \lim_{n\to+\infty} \sum_{k=1}^{n} f\left(m + k \cdot \frac{M+\varepsilon-m}{n}\right)\left(E_{\left(m+k\cdot\frac{M+\varepsilon-m}{n}\right)} - E_{\left(m+(k-1)\cdot\frac{M+\varepsilon-m}{n}\right)}\right)$$

while

$$p_k(S) = \lim_{n\to+\infty} \sum_{k=1}^{n} p_k\left(m + k \cdot \frac{M+\varepsilon-m}{n}\right)\left(E_{\left(m+k\cdot\frac{M+\varepsilon-m}{n}\right)} - E_{\left(m+(k-1)\cdot\frac{M+\varepsilon-m}{n}\right)}\right)$$

i.e.

$$A_f - p_k(S) = A_f - A_{p_k} = A_{f-p_k}$$

Given ε, there is $N \in \mathbb{N}$ for which

$$k \geq N \Longrightarrow \max_{x\in[m,M]} |p_k(x) - f(x)| \leq \varepsilon$$

the inequality proved in (a) shows that

$$k \geq N \Longrightarrow ||p_k(S) - A_f|| \leq \max_{x\in[m,M]} |p_k(x) - f(x)| \leq \varepsilon$$

i.e. A_f is the limit. The equality for $\langle f(S)u, v\rangle$ follows from Exercice 3, Week 5, by taking the limit.