Homework Solution Week 5

1. (a) $m = M = 0$, $E_\lambda = 0$ for $\lambda \leqslant 0$ and $E_\lambda = I$ for $\lambda > 0$.

(b) $m = M = 1$, $E_\lambda = 0$ for $\lambda \leqslant 1$ and $E_\lambda = I$ for $\lambda > 1$.

2. First observe that, for all $\lambda \in \mathbb{R}$,

$$\langle |\cdot - \lambda| u, u\rangle = \int_0^1 |x-\lambda||u(x)|^2 \, \mathrm{d}x \geqslant 0, \quad \forall u \in L^2[0,1].$$

Hence, the operator $T_\lambda : L^2[0,1] \to L^2[0,1]$, $T_\lambda u(x) := |x-\lambda| u(x)$, is positive. Furthermore,

$$T_\lambda^2 u(x) = |x-\lambda|^2 u(x) = (x-\lambda)^2 u(x) = (X - \lambda I)^2 u(x).$$

By uniqueness of the square root, it follows that $T_\lambda = |X - \lambda I|$, for all $\lambda \in \mathbb{R}$.

Now, for all $u \in L^2[0,1]$ and $x \in [0,1]$,

$$[(X-\lambda I) - |X - \lambda I|]\, u(x) = [(x-\lambda) - |x-\lambda|]\, u(x) = -2(x-\lambda)^- u(x).$$

It follows that

$$\ker\,[(X-\lambda I) - |X-\lambda I|] = \begin{cases} L^2[0,1] & \text{if } \lambda \leqslant 0, \\ \{u \in L^2[0,1]\,;\, u(x) = 0, \text{ a.e. } x \leqslant \lambda\} & \text{if } \lambda \in (0,1], \\ \{0\} & \text{if } \lambda > 1. \end{cases}$$

Hence, the projection $E_+(\lambda)$ onto $\ker\,[(X-\lambda I) - |X-\lambda I|]$ is given by

$$E_+(\lambda) u = \begin{cases} u & \text{if } \lambda \leqslant 0, \\ \chi_{(\lambda,1]} u & \text{if } \lambda \in (0,1], \\ 0 & \text{if } \lambda > 1. \end{cases}$$

The result now follows by letting $E_\lambda = I - E_+(\lambda)$.

3. Since p is a real polynomial, we know that $p(S)$ is symmetric and that

$$p(S) = \int_m^{M+\varepsilon} p(\lambda)\, \mathrm{d}E_\lambda.$$

We shall now prove that

$$\langle p(S)u, u\rangle = \int_m^{M+\varepsilon} p(\lambda)\, \mathrm{d}\langle E_\lambda u, u\rangle, \quad \forall u \in \mathcal{H}. \tag{1}$$

We first observe that since the function $\lambda \mapsto \langle E_\lambda u, u\rangle$ is increasing, it belongs to $BV[m, M+\varepsilon]$. Indeed, for any partition of $[m, M+\varepsilon]$, $m = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = M + \varepsilon$, we have

$$\begin{aligned} \sum_{k=1}^n \left|\langle E_{\lambda_k} u, u\rangle - \langle E_{\lambda_{k-1}} u, u\rangle\right| &= \sum_{k=1}^n \langle E_{\lambda_k} u, u\rangle - \langle E_{\lambda_{k-1}} u, u\rangle = \sum_{k=1}^n \big\langle (E_{\lambda_k} - E_{\lambda_{k-1}})u, u\big\rangle \\ &= \Big\langle \sum_{k=1}^n (E_{\lambda_k} - E_{\lambda_{k-1}})u, u\Big\rangle = \big\langle (E_{\lambda_n} - E_{\lambda_0})u, u\big\rangle = \|u\|^2. \end{aligned}$$

Therefore, thanks to Theorem A.2.1, the RHS of (1) is well defined. Now, consider a sequence of partitions $(\Pi_l)_{l=1}^\infty$, denoted as

$$\Pi_l: \quad m = \lambda_0^l < \lambda_1^l < \cdots < \lambda_{n_l-1}^l < \lambda_{n_l}^l = M + \varepsilon, \quad l \geqslant 1.$$

By definition of the Riemann–Stieltjes integral, we have

$$\begin{aligned}\langle p(S)u,u\rangle &= \Big\langle \lim_{l\to\infty}\sum_{k=1}^{n_l} p(\lambda_k^l)(E_{\lambda_k^l}-E_{\lambda_{k-1}^l})u,u\Big\rangle \\ &= \lim_{l\to\infty}\sum_{k=1}^{n_l} p(\lambda_k^l)\big[\big\langle E_{\lambda_k^l}u,u\big\rangle-\big\langle E_{\lambda_{k-1}^l}u,u\big\rangle\big] = \int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\,\langle E_\lambda u,u\rangle\,.\end{aligned}$$

As for the general case, one has

$$\begin{aligned}\langle p(S)u,v\rangle &= \frac14\Big[\ \langle p(S)(u+v),u+v\rangle-\langle p(S)(u-v),u-v\rangle+i\big(\ \langle p(S)(u+iv),u+iv\rangle-\langle p(S)(u-iv),u-iv\rangle\ \big)\Big]. \\ &= \frac14\Big[\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda(u+v),u+v\rangle-\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda(u-v),u-v\rangle \\ &\quad+i\big(\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda(u+iv),(u+iv)\rangle-\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda(u-iv),(u-iv)\rangle\big)\Big]\end{aligned}$$

using the Polarization Identity. Similarly, by Polarization Identity, one notes that

$$\lambda\mapsto\langle E_\lambda u,v\rangle=\frac14\Big[\langle E_\lambda(u+v),u+v\rangle-\langle E_\lambda(u-v),u-v\rangle+i\langle E_\lambda(u+iv),(u+iv)\rangle-i\langle E_\lambda(u-iv),(u-iv)\rangle\Big]$$

is a complex linear combination of monotone functions. Hence, $\lambda\mapsto\langle E_\lambda u,v\rangle$ is a complex-valued function of bounded variation, as well as all the other $\lambda\mapsto\langle E_\lambda(u+v),u+v\rangle$ etc... In particular, $d\langle E_\lambda u,v\rangle$, as well as all the other $d\langle E_\lambda(u+v),u+v\rangle$ give rise to well-defined complex-valued Stieltjes-Riemann integrals (see Appendix A in the lecture notes) and one must have

$$\begin{aligned}&\frac14\Big[\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda(u+v),u+v\rangle-\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda(u-v),u-v\rangle \\ &\quad+i\big(\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda(u+iv),(u+iv)\rangle-\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda(u-iv),(u-iv)\rangle\big)\Big] \\ &\quad=\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda u,v\rangle\end{aligned}$$

using the properties of Stieltjes integration (Properties A.3.1). This leads to the desired equality.

Note that in Appendix B, a Stieltjes-Lebesgue measure μ_ϕ can be defined when ϕ is increasing, e.g. for $\phi(\lambda)=\langle E_\lambda(u+v),u+v\rangle$ etc.... Then we can define

$$\mu\langle E_\lambda u,v\rangle:=\frac14\Big[\mu\langle E_\lambda(u+v),u+v\rangle-\mu\langle E_\lambda(u-v),u-v\rangle+i\big(\mu\langle E_\lambda(u+iv),(u+iv)\rangle-\mu\langle E_\lambda(u-iv),(u-iv)\rangle\big)\Big]$$

as a 'complex-valued measure'. This is only a definition (measure in the usual sense must be nonnegative) and not a property. But when integrating continuous functions, both the Stieltjes-Riemann and the Stieltjes-Lebesgue integral give the same result.

4. Consider two spectral families $(E_\lambda),(F_\lambda)$ satisfying parts (a) to (d) of Spectral Theorem I. We already know that $E_\lambda=F_\lambda$ for all $\lambda<m$ and $\lambda>M$. Fix $u\in\mathcal{H}$. By (1),

$$\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda u,u\rangle=\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle F_\lambda u,u\rangle.$$

As a combination of BV functions, $\phi(\lambda):=\langle E_\lambda u,u\rangle-\langle F_\lambda u,u\rangle$ belongs to $BV[m,M+\varepsilon]$, and we have

$$\int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\phi(\lambda)=0,$$

for any real polynomial p. By the Weierstrass approximation theorem, for any $f\in C^0([m,M+\varepsilon],\mathbb{R})$, there is a sequence of real polynomials converging to f in $C^0([m,M+\varepsilon],\mathbb{R})$. It follows from Theorem A.3.3

that

$$\int_m^{M+\varepsilon} f(\lambda)\,\mathrm{d}\phi(\lambda) = 0, \quad \forall f \in C^0([m, M+\varepsilon], \mathbb{R}).$$

Since ϕ is left-continuous and $\phi(m) = 0$, Theorem A.3.4 implies that $\phi \equiv 0$ on $[m, M+\varepsilon]$. It follows that

$$\langle (E_\lambda - F_\lambda)u, u\rangle = 0, \quad \forall \lambda \in \mathbb{R}.$$

Since this holds true for any $u \in \mathcal{H}$, we conclude that $E_\lambda = F_\lambda$, for all $\lambda \in \mathbb{R}$.