Homework Solution Week 4

1. We show that $|\lambda| > \|A\| \Rightarrow \lambda \in \rho(A)$. Suppose $|\lambda| > \|A\|$. We observe that

$$(A-\lambda I)u = v \iff u = \frac{1}{\lambda}(Au - v) =: T_v u.$$

Moreover, for all $u_1, u_2 \in \mathcal{H}$,

$$\|T_v u_1 - T_v u_2\| = \frac{1}{|\lambda|}\|A(u_1-u_2)\| \leqslant \frac{\|A\|}{|\lambda|}\|u_1 - u_2\|.$$

Since $\|A\|/|\lambda| < 1$, it follows that T_v is a contraction. Hence, $T_v u = u$ has unique solution, for any $v \in \mathcal{H}$. That is, $\lambda \in \rho(A)$.

2. Let $T = AA^*$. Since A is normal we have, for any $n \in \mathbb{N}$,

$$T^* = T, \quad \|T\| = \|A\|^2, \quad (AA^*)^n = A^n(A^*)^n, \quad (A^*)^n = (A^n)^*.$$

Using these properties, we find that

$$\|T^n\|^{\frac{1}{n}} = \|(AA^*)^n\|^{\frac{1}{n}} = \|A^n(A^*)^n\|^{\frac{1}{n}} = \|A^n(A^n)^*\|^{\frac{1}{n}} = \|A^n\|^{\frac{2}{n}},$$

hence

$$r_\sigma(T) = \lim_{n\to\infty}\|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty}\|A^n\|^{\frac{2}{n}} = r_\sigma(A)^2. \tag{1}$$

On the other hand, since $T = T^*$ we obtain, for any $m \in \mathbb{N}$,

$$\left\|T^{2^m}\right\|^{1/2^m} = \left\|T^{2^{m-1}}T^{2^{m-1}}\right\|^{1/2^m} = \left\|T^{2^{m-1}}(T^{2^{m-1}})^*\right\|^{1/2^m} = \left\|T^{2^{m-1}}\right\|^{1/2^{m-1}}.$$

It follows by induction that $\left\|T^{2^m}\right\|^{1/2^m} = \|T\|$ for any $m \in \mathbb{N}$. Hence,

$$r_\sigma(T) = \|T\| = \|A\|^2. \tag{2}$$

The result follows from (1) and (2).

3. We verify the three properties of a partial order.

<u>Reflexivity:</u> $A - A \geqslant 0$ is trivial.

<u>Transitivity:</u> If $A \geqslant B$ and $B \geqslant C$ we have, for all $u \in \mathcal{H}$,

$$\langle (A-C)u, u\rangle = \langle (A-B)u,u\rangle + \langle (B-C)u,u\rangle \geqslant 0,$$

which shows that $A \geqslant C$.

<u>Antisymmetry:</u> Suppose $A \geqslant B$ and $B \geqslant A$. Then, $\langle (A-B)u,u\rangle \geqslant 0$ and $\langle (B-A)u,u\rangle \geqslant 0$, for all $u \in \mathcal{H}$. It follows that $\langle (A-B)u,u\rangle = 0$, for all $u \in \mathcal{H}$. Hence,

$$\|A-B\| = \sup_{\|u\|=1}|\langle (A-B)u,u\rangle| = 0,$$

which shows that $A = B$.

4. (a) We have $\|Au\| = \|u\|$, for all $u \in \ell^2$. Hence, $\|A\| = 1$.

(b) A^* is the double left shift $(u_1, u_2, u_3, \dots) \mapsto (u_3, \dots)$.

(c) Arguments similar to Problem 5, Week 2, show that

$$\sigma_p(A) = D_1(0),\ \sigma_r(A) = \emptyset,\ \sigma_c(A) = \mathbb{S}^1,$$
$$\sigma_p(A^*) = \emptyset,\ \sigma_r(A) = D_1(0),\ \sigma_c(A) = \mathbb{S}^1.$$

(d) Obviously, $A = T^2$, where T is the right shift introduced in Week 2. However, A is not positive since $\langle Au, u\rangle = -1$ for $u = (1,1,-1,0,0,\dots)$.

5. (a) For all $u \in L^2[0,1]$,

$$\|Au\|_{L^2}^2 = \int_{[0,1]} |a(x)u(x)|^2 \,\mathrm{d}x \leqslant \|a\|_{L^\infty}^2 \|u\|_{L^2}^2,$$

hence A is bounded and $\|A\| \leqslant \|a\|_{L^\infty}$.

On the other hand, taking $\varepsilon > 0$ and $E = \{x \in [0,1]\,;\, |a(x)| \geqslant \|a\|_{L^\infty} - \varepsilon\}$, it follows by definition of $\|a\|_{L^\infty}$ that $|E| > 0$. Letting $u(x) = |E|^{-1/2}\chi_E(x)$, we obtain $\|u\| = 1$ and

$$\|Au\|_{L^2}^2 = |E|^{-1} \int_E |a(x)|^2 \,\mathrm{d}x \geqslant (\|a\|_{L^\infty} - \varepsilon)^2.$$

Since this inequality holds for any $\varepsilon > 0$, we conclude that $\|A\| = \|a\|_{L^\infty}$.

(b) If $a \geqslant 0$ a.e., A is positive and its square root is the operator $S : L^2[0,1] \to L^2[0,1]$ defined by $(Su)(x) = \sqrt{a(x)}u(x)$, a.e. $x \in [0,1]$.

(c) (i) Seeking $u \in L^2[0,1] \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that $Au = \lambda u$ amounts to finding $\lambda \in \mathbb{C}$ and $u \in L^2[0,1]$, such that

$$[a(x) - \lambda]u(x) = 0, \quad \text{a.e. } x \in [0,1] \quad \text{and} \quad u(x) \not\equiv 0, \quad \text{a.e. } x \in [0,1].$$

It follows that $\sigma_p(A) = \big\{\lambda \in \mathbb{C}\,;\, |\{a(x) = \lambda\}| > 0\big\}$.

(ii) If $\lambda \in \sigma_r(A)$, then λ satisfies $\lambda \notin \sigma_p(A)$ and $\bar{\lambda} \in \sigma_p(A^*)$. Since $(A^*u)(x) = \overline{a(x)}u(x)$, this means that $|\{a(x) = \lambda\}| = 0$ and $|\{\overline{a(x)} = \bar{\lambda}\}| > 0$, a contradiction. Hence, $\sigma_r(A) = \emptyset$.

(iii) We show that

(A) $\exists \varepsilon > 0$ s.t. $|\{|a(x) - \lambda| < \varepsilon\}| = 0 \implies \lambda \in \rho(A)$;
(B) $\forall \varepsilon > 0,\ |\{|a(x) - \lambda| < \varepsilon\}| > 0 \implies \lambda \notin \rho(A)$.

Firstly, given any $v \in L^2[0,1]$ and $\lambda \notin \sigma_p(A)$, solving $(A - \lambda)u = v$ for $u \in L^2[0,1]$ yields

$$u(x) = \frac{1}{a(x) - \lambda} v(x), \text{ a.e. } x \in [0,1]. \tag{3}$$

(A) Let $\varepsilon > 0$ such that $|a(x) - \lambda| \geqslant \varepsilon$, a.e. $x \in [0,1]$. In particular, $\lambda \notin \sigma_p(A)$. Then (3) defines on $L^2[0,1]$ a bounded multiplication operator $v \mapsto (A - \lambda)^{-1}v$, with

$$\big\|(A - \lambda)^{-1}v\big\|^2 = \left\|\frac{1}{a(\cdot) - \lambda} v\right\|^2 = \int_0^1 \frac{1}{|a(x) - \lambda|^2}|v(x)|^2 \,\mathrm{d}x \leqslant \varepsilon^{-2}\,\|v\|^2 \implies \big\|(A - \lambda)^{-1}\big\| \leqslant \varepsilon^{-1}.$$

Hence, $\lambda \in \rho(A)$.

(B) Taking $\varepsilon = 1/n$, we deduce from the assumption: for all $n \geqslant 1$, there exists $\Omega_n \subset [0,1]$ such that $|\Omega_n| > 0$ and

$$|a(x) - \lambda| < \frac{1}{n}, \quad \forall x \in [0,1].$$

Consider now $v_n = \chi_{\Omega_n}$, $n \geqslant 1$. It follows that

$$\|v_n\|^2 = |\Omega_n| \quad \text{and} \quad \big\|(A - \lambda)^{-1}v_n\big\|^2 = \int_{\Omega_n} \frac{1}{|a(x) - \lambda|^2} |\,\mathrm{d}x > n^2|\Omega_n|, \quad \forall n \geqslant 1.$$

Hence,

$$\frac{\big\|(A - \lambda)^{-1}v_n\big\|}{\|v_n\|} > n, \quad \forall n \geqslant 1.$$

This shows that $(A - \lambda)^{-1}$ is unbounded, and thus $\lambda \notin \rho(A)$.