Homework Solution Week 3

1. (a) Let $u \neq 0$ and $\lambda \in \mathbb{C}$ satisfy $Su = \lambda u$. Then,

$$\lambda \|u\|^2 = \langle \lambda u, u\rangle = \langle Su, u\rangle = \langle u, Su\rangle = \langle u, \lambda u\rangle = \bar{\lambda}\|u\|^2 \implies \lambda = \bar{\lambda}.$$

Furthermore, if $\mu \neq \lambda$ is another eigenvalue, let $v \neq 0$ be a corresponding eigenvector. Then, since $\mu \in \mathbb{R}$,

$$\langle u, v\rangle = \frac{1}{\lambda-\mu}(\lambda-\mu)\langle u, v\rangle = \frac{1}{\lambda-\mu}(\langle \lambda u, v\rangle - \langle u, \mu v\rangle) = \frac{1}{\lambda-\mu}(\langle Su, v\rangle - \langle u, Sv\rangle) = 0.$$

(b1) First suppose that $\lambda \in \rho(S)$. Since $\mathcal{H}$ is a Banach space, it follows that $(S-\lambda I)^{-1}$ is well defined and bounded on $\mathcal{H}$. Consider $u \in \mathcal{H}$ and $v = (S-\lambda I)^{-1}u$. We have

$$\begin{aligned}\|u\| = \|(S-\lambda I)v\| = \left\|(S-\lambda I)(S-\lambda I)^{-1}u\right\| &= \left\|(S-\lambda I)^{-1}(S-\lambda I)u\right\| \\ &\leqslant \left\|(S-\lambda I)^{-1}\right\| \|(S-\lambda I)u\|.\end{aligned}$$

Hence, $\|(S-\lambda I)u\| \geqslant \left\|(S-\lambda I)^{-1}\right\|^{-1}\|u\|$, for all $u \in \mathcal{H}$.

Conversely, suppose there exists $C > 0$ such that $\|(S-\lambda I)u\| \geqslant C\|u\|$, for all $u \in \mathcal{H}$. It follows that $\ker(S-\lambda I) = \{0\}$. If $\lambda \in \mathbb{R}$, then $(S-\lambda I)^* = S-\lambda I$ and $\overline{\operatorname{rge}(S-\lambda I)} = \mathcal{H}$ (Exercise Sheet 2, Exercise 4). If $\lambda \notin \mathbb{R}$, it follows from 1.(a) that $\lambda, \overline{\lambda}$ are not eigenvalues. Hence, $\ker(S-\lambda I) = \ker(S-\overline{\lambda}I) = \{0\}$. As $(S-\lambda I)^* = S-\overline{\lambda}I$, we deduce $\overline{\operatorname{rge}(S-\lambda I)} = \mathcal{H}$ (Exercise Sheet 2, Exercise 4) as well. Hence, either $\lambda \in \rho(S)$ or $\lambda \in \sigma_c(S)$. To rule out $\lambda \in \sigma_c(S)$, we show that $\operatorname{rge}(S-\lambda I)$ is closed, hence $\operatorname{rge}(S-\lambda I) = \mathcal{H}$. Let $(y_n) \subset \operatorname{rge}(S-\lambda I)$ be a convergent sequence in $\mathcal{H}$, $\lim\limits_{n\to+\infty} y_n = y \in \mathcal{H}$. Then $y_n = (S-\lambda I)x_n$ for some $x_n \in \mathcal{H}$ and

$$\|y_n - y_m\| = \|(S-\lambda I)(x_n - x_m)\| \geqslant C\|x_n - x_m\|,$$

showing that (x_n) is a Cauchy sequence in $\mathcal{H}$. Let $x = \lim\limits_{n\to+\infty} x_n$. It follows by continuity that $y = \lim(S-\lambda I)x_n = (S-\lambda I)x \in \operatorname{rge}(S-\lambda I)$. Hence, $\lambda \in \rho(S)$.

(b2) Trivially equivalent to (b1).

(c) Write $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_2 \neq 0$ and let $u \in \mathcal{H}$. A direct calculation shows that

$$\|(S-\lambda I)u\|^2 = \|(S-\lambda_1 I)u\|^2 + |\lambda_2|^2\|u\|^2 \geqslant |\lambda_2|^2\|u\|^2.$$

Hence, the result follows from (b1).

(d1) Suppose $\lambda > M$. Since

$$\begin{aligned}&-\langle (S-\lambda I)u, u\rangle = -\langle Su, u\rangle + \lambda\langle u, u\rangle \geqslant (\lambda - M)\|u\|^2 \\ &\implies (\lambda-M)\|u\|^2 \leqslant |\langle (S-\lambda I)u, u\rangle| \leqslant \|(S-\lambda I)u\|\|u\|, \quad \forall u \in \mathcal{H},\end{aligned}$$

we deduce from (b1) that $\lambda \in \rho(S)$, hence $(M, \infty) \subset \rho(S)$. A similar argument shows that $(-\infty, m) \subset \rho(S)$.

(d2) Firstly, if S is positive, we have $\|S\| = \sup_{\|u\|=1}\langle Su, u\rangle = M \geqslant m \geqslant 0$. By definition of M, there exists a sequence $(u_n) \subset \mathcal{H}$, with $\|u_n\| = 1$, such that $\langle Su_n, u_n\rangle = M - \delta_n$, with $\delta_n \geqslant 0,\ \delta_n \to 0$. Then

$$\begin{aligned}\|Su_n - Mu_n\|^2 &= \|Su_n\|^2 + M^2\|u_n\|^2 - 2M\langle Su_n, u_n\rangle \\ &\leqslant M^2 + M^2 - 2M(M-\delta_n) \to 0 \quad (n\to\infty).\end{aligned}$$

We conclude by (b2) that $M \in \sigma(S)$.

On the other hand, if S is negative, we have $\|S\| = \sup_{\|u\|=1} |\langle Su, u\rangle| = |m| \geqslant |M|$. By definition of m, there exists a sequence $(v_n) \subset \mathcal{H}$, with $\|v_n\| = 1$, such that $\langle Sv_n, v_n\rangle = m + \varepsilon_n$, with $\varepsilon_n \geqslant 0,\ \varepsilon_n \to 0$. Then

$$\begin{aligned}\|Su_n - mu_n\|^2 &= \|Su_n\|^2 + m^2\|u_n\|^2 - 2m\langle Su_n, u_n\rangle \\ &\leqslant m^2 + m^2 - 2m(m+\varepsilon_n) \to 0 \quad (n \to \infty).\end{aligned}$$

The general case is treated as follows. Let $\gamma > \max\{|m|, |M|\}$. To prove that $M \in \sigma(S)$, observe that $m+\gamma, M+\gamma$ are respectively the lower and upper bound of the positive symmetric operator $S + \gamma I$. Then the previous argument can be used to show that

$$\|Su_n - Mu_n\|^2 = \|(S+\gamma I)u_n - (M+\gamma)u_n\|^2 \to 0.$$

Similarly, one reduces the proof of $m \in \sigma(S)$ to the case of a negative operator by considering $S - \gamma I$.

2. For any $n \geqslant 1$, we have

$$\langle S_n u, v\rangle = \langle u, S_n v\rangle, \quad \forall u, v \in \mathcal{H}.$$

By continuity of the inner product, the result follows directly by letting $n \to \infty$.

3. Consider $\mathcal{H} = \ell^2(\mathbb{C})$ and the left-shift operator applied m-times $A_m((u_1, u_2, ...)) = (u_{m+1}, u_{m+2}, ...)$ i.e. $(A_m(u))_i = u_{m+i}$ for $i \geq 1$. Then

$$\lim_{m\to+\infty} A_m(u) = 0 \quad \forall u \in \mathcal{H}$$

because

$$\lim_{m\to+\infty} ||A_m(u) - 0|| = \lim_{m\to+\infty} \sum_{i=m+1}^{+\infty} |u_i|^2 = 0 \quad \forall u \in \mathcal{H}$$

while $||A_m|| = 1$ for all $m \geq 1$. Hence, A_m converges strongly (i.e. "pointwisely") to zero, but not in operator norm.

4. (a) $BS = SB \implies SB^* = (BS)^* = (SB)^* = B^*S$.

(b) If both S and B are symmetric, we have

$$SB \text{ symmetric} \iff (SB)^* = SB \iff B^*S^* = SB \iff BS = SB.$$

(c) It follows from (b) that $S^n = S \circ ... \circ S$ is symmetric for any $n \geq 1$ and $S^0 = Id$ is symmetric as well. Moreover, if A and $B \in \mathcal{B}(\mathcal{H})$ are symmetric and if $\lambda, \mu \in \mathbb{R}$, then

$$\langle(\lambda A + \mu B)u, v\rangle = \lambda\langle Au, v\rangle + \mu\langle Bu, v\rangle = \lambda\langle u, A^*v\rangle + \mu\langle u, B^*v\rangle = \langle u, (\lambda A^* + \mu B^*)v\rangle \quad \forall u, v \in \mathcal{H}$$

so $(\lambda A + \mu B)^* = \lambda A^* + \mu B^*$ by uniqueness of the adjoint operator. If $P(x) = a_n x^n + ... + a_1 x + a_0$ is a real polynomial, then $P(S) = a_n S^n + ... + a_1 S + a_0 Id$ is symmetric given what we've just proved.

5. Firstly, for all $u \in \mathcal{H}$, $\|Xu\|^2 = \int_{[0,1]} |x|^2 |u(x)|^2\,\mathrm{d}x \leqslant \int_{[0,1]} |u(x)|^2 = \|u\|^2$, hence $X \in \mathcal{B}(\mathcal{H})$ and $\|X\| \leqslant 1$. Furthermore, for all $u, v \in \mathcal{H}$,

$$\langle Xu, v\rangle = \int_{[0,1]} xu(x)\overline{v(x)}\,\mathrm{d}x = \int_{[0,1]} u(x)\overline{xv(x)}\,\mathrm{d}x = \langle u, Xv\rangle,$$

showing that X is symmetric.

Next, for $u \in L^2[0,1]$, $Xu = \lambda u \Rightarrow |x - \lambda||u(x)| = 0$ a.e. $x \in [0,1] \Rightarrow u = 0$, so X has no eigenvalues. Since X is symmetric, we also have $\sigma_r(X) = \emptyset$, so $\sigma(X) = \sigma_c(X)$.

We will now prove that $\sigma(X) = [0,1]$. Firstly, if $\lambda \notin [0,1]$, the equation $(X - \lambda I)u = v$ yields

$$u(x) = \frac{1}{x-\lambda}v(x), \quad \|u\|^2 = \left\|(X-\lambda I)^{-1}v\right\|^2 = \int_{[0,1]} \frac{|v(x)|^2}{|x-\lambda|^2}\,\mathrm{d}x \leqslant \delta_\lambda^{-2}\|v\|^2,$$

where $\delta_\lambda > 0$ is the distance from λ to $[0,1]$. Hence, $\lambda \in \rho(X)$ and $\left\|(X-\lambda I)^{-1}\right\| \leqslant \delta_\lambda^{-1}$. This shows that $\sigma(X) \subset [0,1]$.

To prove the converse inclusion, let $\lambda \in [0,1]$. For simplicity, we suppose that $\lambda \in (0,1)$, but the argument we present can easily be adapted to the end-points. Since $\sigma(X) = \sigma_c(X)$, either $\lambda \in \rho(X)$, $\mathrm{rge}(X - \lambda I)$ is dense and $(X - \lambda I)^{-1}$ is bounded, or $\lambda \in \sigma_c(X)$, i.e. $\mathrm{rge}(X - \lambda I)$ is dense but $(X - \lambda I)^{-1}$ is unbounded. So we only need to show that $(X - \lambda I)^{-1}$ is unbounded. To this aim, let $u_n \in L^2[0,1]$ be the characteristic function of the set $[0,1] \setminus [\lambda - 1/n, \lambda + 1/n]$ (for n large enough). We clearly have that $\|u_n\| \leqslant 1$. On the other hand,

$$\left\|(X - \lambda I)^{-1} u_n\right\|^2 = \int_{[0,1]\setminus[\lambda-1/n,\lambda+1/n]} \frac{1}{|x - \lambda|^2}\,\mathrm{d}x$$

is finite for all n (large enough), but $\left\|(X - \lambda I)^{-1} u_n\right\| \to \infty$ as $n \to \infty$, showing that $(X - \lambda I)^{-1}$ is indeed unbounded. Hence, $\lambda \in \sigma(X)$, and we conclude that $\sigma(X) = [0,1]$.

Remarks:

(i) Another way to prove that $\lambda \in [0,1] \Rightarrow \lambda \in \sigma(X)$ is as follows. Let $\lambda \in [0,1]$. Suppose by contradiction that $\lambda \in \rho(X)$. Since $\mathcal{H}$ is a Banach space, we know from Problem 4, Week 1, that $\mathrm{rge}(X - \lambda I) = \mathcal{H}$. But this is not true, as $v \equiv 1 \notin \mathrm{rge}(X - \lambda I)$, since $u(x) = (x - \lambda)^{-1} \notin L^2[0,1]$.

(ii) Even though $\mathrm{rge}(X - \lambda I) \neq \mathcal{H}$, we can see explicitly here that $\overline{\mathrm{rge}(X - \lambda I)} = \mathcal{H}$. For any $v \in L^2[0,1]$, consider $\varepsilon > 0$ and v_ε defined by $v_\varepsilon(x) = v(x)$ for $x \notin (\lambda - \varepsilon, \lambda + \varepsilon)$ and $v_\varepsilon(x) = 0$ for $x \in (\lambda - \varepsilon, \lambda + \varepsilon)$. One can show two things : $v_\varepsilon \in \mathrm{rge}(X - \lambda I)$, meaning that $(X - \lambda I)u = v_\varepsilon$ can be solved in $L^2[0,1]$ for any $\varepsilon > 0$. Second, v_ε can be made as close as we want to v in $L^2[0,1]$ when $\varepsilon \to 0^+$.