Homework Solution Week 2

1. (a) Write $u = Pu + u',\ v = Pv + v'$, with $u', v' \in M^\perp$. Then

$$\langle Pu, v\rangle = \left\langle Pu, Pv + v'\right\rangle = \langle Pu, Pv\rangle = \left\langle Pu + u', Pv\right\rangle = \langle u, Pv\rangle\,.$$

(b) First, $\operatorname{rge} P \subset M$ by definition of P. To show that $M \subset \operatorname{rge} P$, consider $u \in M$. Then, for any $u' \in M^\perp$, we have $P(u + u') = Pu = u$, hence $u \in \operatorname{rge} P$.

Next, $M^\perp \subset \ker P$ by definition of P. To show that $\ker P \subset M^\perp$, take $u \in \ker P$ and write it as $u = Pu + u'$, with $u' \in M^\perp$. Then, $Pu = 0 \Rightarrow u = u' \in M^\perp$.

(c) Writing the unique decomposition $u = Pu + u'$, with $u' \in M^\perp$, we see that indeed $(I - P)u = u'$, the second component of u.

(d) We first suppose that $M \subset N$ and prove that $PQ = QP = P$. Consider $u \in \mathcal{H}$. Write $u = u_M + u_{M^\perp}$ the projection into M and $M^\perp$ and further project the perpendicular part into $N, N^\perp$, i.e.

$$u = u_M + u_{M^\perp} = u_M + (u_{M^\perp})_N + (u_{M^\perp})_{N^\perp}$$

Observe that $(u_{M^\perp})_N \in M^\perp$. Otherwise, one can further decompose

$$u = u_M + u_{M^\perp} = \underbrace{u_M + [(u_{M^\perp})_N]_M}_{\in M} + \underbrace{[(u_{M^\perp})_N]_{M^\perp} + (u_{M^\perp})_{N^\perp}}_{\in M^\perp}$$

By uniqueness of the orthogonal projection,

$$u_M = u_M + [(u_{M^\perp})_N]_M \Rightarrow [(u_{M^\perp})_N]_M = 0$$

It follows that

$$PQu = PQ\left(u_M + (u_{M^\perp})_N + (u_{M^\perp})_{N^\perp}\right) = P\left(u_M + (u_{M^\perp})_N\right) = u_M = Pu$$

Similarly,

$$QPu = QP\left(u_M + u_{M^\perp}\right) = Q\left(u_M\right) = u_M = Pu$$

Conversely, suppose that $PQ = QP = P$. Then, for any $u \in M$, we have

$$QPu = Pu = u,$$

showing that $u \in \operatorname{rge} Q = N$. Hence, $M \subset N$.

(e) $PQ = 0 \iff \langle PQu, v\rangle = 0\ \forall u, v \in \mathcal{H} \iff \langle Qu, Pv\rangle = 0\ \forall u, v \in \mathcal{H} \iff \operatorname{Im} P \perp \operatorname{Im} Q \iff \langle QPu, v\rangle = 0\ \forall u, v \in \mathcal{H}$.

(f) First suppose $M \subset N$. For $u \in \mathcal{H}$, write $u = v + v'$, with $v \in N$ and $v' \in N^\perp \subset M^\perp$. Then, by (d),

$$\langle Pu, u\rangle = \left\langle Pv, v + v'\right\rangle = \langle Pv, v\rangle\,, \quad \langle Qu, u\rangle = \left\langle v, v + v'\right\rangle = \langle v, v\rangle = \|v\|^2\,.$$

It follows that

$$\langle Pu, u\rangle - \langle Qu, u\rangle = \langle Pv, v\rangle - \|v\|^2 \leqslant 0.$$

We prove the converse by contraposition. Suppose that $M \not\subset N$. Then there exists $u \in M \setminus N$. It follows that $(I - Q)u \neq 0$. Hence,

$$\|u\|^2 = \|Qu\|^2 + \|(I - Q)u\|^2 > \|Qu\|^2 = \langle Qu, Qu\rangle = \langle Qu, u\rangle\,.$$

On the other hand, since $u \in M$, we have $\|u\|^2 = \langle Pu, Pu\rangle = \langle Pu, u\rangle$. We deduce that $\langle Pu, u\rangle > \langle Qu, u\rangle$, as expected.

2. That any projection is symmetric and idempotent is trivial.

Conversely, suppose that $P : \mathcal{H} \to \mathcal{H}$ is bounded, symmetric and idempotent. We will prove that P is the orthogonal projection onto $M := \operatorname{rge} P$.

Firstly, M is a closed subspace. Indeed, suppose $Pu_n \to v \in \mathcal{H}$, for some sequence $(u_n) \subset \mathcal{H}$. Then, by continuity and idempotence, we find that

$$Pv = \lim_{n\to\infty} PPu_n = \lim_{n\to\infty} Pu_n = v,$$

showing that $v \in M$.

Next, let $u \in \mathcal{H}$. By the symmetry of P,

$$\forall v \in M, \quad \langle u - Pu, v\rangle = \langle u, v\rangle - \langle Pu, v\rangle = \langle u, v\rangle - \langle u, Pv\rangle = \langle u, v\rangle - \langle u, v\rangle = 0.$$

This shows that $u - Pu \in M^\perp$ for all $u \in \mathcal{H}$. By uniqueness of the orthogonal decomposition, it follows that P is indeed the orthogonal projection onto M.

3. Writing $u = v + \lambda w$, we have by hypothesis that

$$0 = \langle T(v + \lambda w), v + \lambda w\rangle = \lambda \langle Tw, v\rangle + \bar{\lambda} \langle Tv, w\rangle .$$

Using the special values $\lambda = i$ and $\lambda = 1$, we obtain

$$\langle Tv, w\rangle = \langle Tw, v\rangle , \quad \forall v, w \in \mathcal{H}, \quad \text{and} \quad \langle Tv, w\rangle = -\langle Tw, v\rangle , \quad \forall v, w \in \mathcal{H},$$

respectively. It follows that $\langle Tv, w\rangle = 0$, for all $v, w \in \mathcal{H}$. Letting $w = Tv$, we deduce that $\|Tv\|^2 = 0$, for all $v \in \mathcal{H}$, hence $T = 0$.

A counterexample in the real case is given by $\mathcal{H} = \mathbb{R}^2$ and T a rotation by $\pi/2$ around 0.

4. Since $\ker A^*$ is a closed subspace of M, it suffices to show that $[\ker A^*]^\perp = \overline{\operatorname{rge} A}$. To do so, we deduce from

$$\langle Au, v\rangle = \langle u, A^* v\rangle , \quad \forall u, v \in \mathcal{H},$$

that $v \in \operatorname{rge} A^\perp$ if and only if $v \in \ker A^*$. Hence, using the hint,

$$\operatorname{rge} A^\perp = \ker A^* \implies \overline{\operatorname{rge} A} = [\operatorname{rge} A^\perp]^\perp = [\ker A^*]^\perp.$$

5. (a) For $u = (u_1, u_2, \dots) \in \ell^2$, we have

$$\|Su\|^2 = \sum_{n\geqslant 2} |u_n|^2 = \|u\|^2 - |u_1|^2 \leqslant \|u\|^2 .$$

Since $\|S(0, 1, 0, 0, \dots)\| = 1$, it follows that $\|S\| = 1$. Furthermore, $\|Tu\|^2 = \|u\|^2$, hence $\|T\| = 1$.

(b) We have

$$\langle Su, v\rangle = \sum_{n\geqslant 1} u_{n+1}\overline{v_n} = \sum_{n\geqslant 2} u_n\overline{v_{n-1}} = \langle u, Tv\rangle , \quad \forall u, v \in \mathcal{H}.$$

Hence, $S^* = T$ and $T^* = S^{**} = S$.

(c) First, solving $Su = \lambda u$, we find that $\lambda \in \mathbb{C}$ is an eigenvalue with eigenvector $u = (u_1, u_2, \dots) \in \ell^2$ if $|\lambda| < 1$ and $u_{n+1} = \lambda^n u_1,\ n \geqslant 1,\ u_1 \neq 0$. Since $|\lambda| = 1 \Rightarrow u \notin \ell^2$, it follows that

$$\sigma_p(S) = \{\lambda \in \mathbb{C}\,;\ |\lambda| < 1\} =: D_1(0).$$

On the other hand, $Tu = \lambda u$ yields $u_{n-1} = \lambda u_n,\ n \geqslant 2,\ 0 = \lambda u_1$. It follows that $\lambda = 0$ or $u_1 = 0$. In both cases we find $u = 0$, hence $\sigma_p(T) = \emptyset$.

Now, since $S^* = T$ and $T^* = S$, we deduce that

$$\sigma_r(S) = \emptyset, \quad \sigma_r(T) = D_1(0).$$

Furthermore, since $\|S\| = \|T\| = 1$, $\sigma(S)$ and $\sigma(T)$ are compact subsets of $\overline{D_1(0)}$. It follows that

$$\overline{D_1(0)} = \overline{\sigma_p(S)} \subset \sigma(S) \subset \overline{D_1(0)},$$

hence $\sigma(S) = \overline{D_1(0)}$, and we conclude that $\sigma_c(S) = \sigma(S) \setminus \sigma_p(S) \cup \sigma_r(S) = \partial D_1(0) = \mathbb{S}^1$. Finally, it follows similarly from $\overline{D_1(0)} = \overline{\sigma_r(T)} \subset \sigma(T) \subset \overline{D_1(0)}$ that $\sigma(T) = \overline{D_1(0)}$ and $\sigma_c(T) = \mathbb{S}^1$.