Homework Solution Week 1

1. Let $v \in \mathcal{H}$ and $u_n \to u$ in $\mathcal{H}$, $n \to \infty$. Then, by Cauchy-Schwarz,

$$|\langle u_n, v\rangle - \langle u, v\rangle| = |\langle u_n - u, v\rangle| \leqslant \|u_n - u\| \|v\| \to 0, \quad n \to \infty,$$

showing that $u \mapsto \langle u, v\rangle$ is continuous. That $v \mapsto \langle u, v\rangle$ is also continuous now follows from this result and the identity $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

On the other hand, if $u_n \to u$ in $\mathcal{H}$, $n \to \infty$, the triangle inequality implies that

$$|\,\|u_n\| - \|u\|\,| \leqslant \|u_n - u\| \to 0, \quad n \to \infty,$$

showing that $u \mapsto \|u\|$ is continuous.

2. We define

$$\|A\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} := \inf \Omega_A, \quad \Omega_A := \{C \geqslant 0\,;\, \|Au\|_{\mathcal{H}_2} \leqslant C\,\|u\|_{\mathcal{H}_1} \ \ \forall u \in \mathcal{H}_1\}.$$

On the other hand, we let

$$S_A := \sup_{u \in \mathcal{H}_1 \setminus \{0\}} \frac{\|Au\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_1}}.$$

We will prove that $\|A\| = S_A$. We drop the indices from the norms to simplify the notation.

Firstly, by definition of S_A, we have that $\|Au\| \leqslant S_A \|u\|$ for all $u \in \mathcal{H}_1$, hence $S_A \in \Omega_A$ and $S_A \geqslant \|A\|$. Now suppose by contradiction that $S_A > \|A\|$. Then, by definition of $\|A\|$, there exists $C \in [\|A\|, S_A)$ such that $\|Au\| \leqslant C\,\|u\|$ for all $u \in \mathcal{H}_1$. Hence,

$$\frac{\|Au\|}{\|u\|} \leqslant C < S_A \quad \forall u \in \mathcal{H}_1 \setminus \{0\}.$$

But this contradicts the definition of S_A.

Finally, we prove that

$$\sup_{u \neq 0} \frac{\|Au\|}{\|u\|} = \sup_{\|u\|=1} \|Au\| = \sup_{\|u\| \leqslant 1} \|Au\|.$$

The first identity follows directly by the linearity of A and the positive homogeneity of the norm. We now prove the second one. The inequality $\sup_{\|u\|=1} \|Au\| \leqslant \sup_{\|u\|\leqslant 1} \|Au\|$ is trivial. To conclude, observe that

$$\sup_{\|u\|=1} \|Au\| = \sup_{u \neq 0} \frac{\|Au\|}{\|u\|} \geqslant \sup_{0<\|u\|\leqslant 1} \frac{\|Au\|}{\|u\|} \geqslant \sup_{\|u\|\leqslant 1} \|Au\|.$$

3. We recall that a linear operator is bounded if and only if it is continuous.

Suppose that A is bounded. Consider a sequence $((u_n, v_n)) \subset \boldsymbol{G}_A$ such that $(u_n, v_n) \to (u, v)$ as $n \to \infty$. By continuity of A, $v_n = Au_n \to Au$, hence $(u, v) = (u, Au)$. Therefore, $(u, v) \in \boldsymbol{G}_A$, showing that $\boldsymbol{G}_A$ is closed.

Conversely, suppose that $\boldsymbol{G}_A \subset \mathcal{X}_1 \times \mathcal{X}_2$ is closed. Then $\boldsymbol{G}_A$, endowed with the product norm inherited from $\mathcal{X}_1 \times \mathcal{X}_2$, is a Banach space. Consider the projections $\pi_j : \boldsymbol{G}_A \to \mathcal{X}_j$ $(j = 1, 2)$ defined by

$$\pi_1 : (u, Au) \mapsto u, \quad \pi_2 : (u, Au) \mapsto Au.$$

Both are bounded linear maps of norm one. Since π_1 is a bounded bijection between Banach spaces, the inverse mapping theorem ensures that π_1^{-1} is continuous. Since, $A = \pi_2 \circ \pi_1^{-1}$, we conclude that A is continuous, hence bounded.

4. By assumption, $R_\lambda(A) = (A - \lambda I)^{-1}$ is well defined and bounded on a dense subspace $\mathfrak{D}_\lambda(A)$ of $\mathcal{X}$. Let $u \in \mathcal{X}$. The Closed Graph Theorem implies that the graph

$$\boldsymbol{G}_{(A-\lambda I)} = \{(x, (A - \lambda I)x) : x \in X\} \subset X \times X$$

of $(A-\lambda I)$ is closed. Hence,

$$\boldsymbol{G}_{R_\lambda(A)} = \{(x,y) \in X \times X : (y,x) \in \boldsymbol{G}_{(A-\lambda I)}\}$$

is closed as well. We prove that $\mathfrak{D}_\lambda(A)$ is closed, hence $\mathfrak{D}_\lambda(A) = \overline{\mathfrak{D}_\lambda(A)} = X$. Take any sequence $(u_n) \subset \mathfrak{D}_\lambda(A)$ such that $u_n \to u \in X$. We need to show that $u \in \mathfrak{D}_\lambda(A)$. Observe that

$$\|R_\lambda(A)u_n - R_\lambda(A)u_m\| \leqslant \|R_\lambda(A)\| \, \|u_n - u_m\| \to 0 \quad (n, m \to \infty),$$

i.e. $(R_\lambda(A)u_n)$ is a Cauchy sequence in $\mathcal{X}$, hence converges. Call v its limit. We have proved that $(u_n, R_\lambda(A)u_n) \subset \boldsymbol{G}_{R_\lambda(A)}$ converges to $(u,v) \in X$. Since the graph of $R_\lambda(A)$ is closed, $(u,v) \in \boldsymbol{G}_{R_\lambda(A)}$. In particular, $u \in \mathfrak{D}_\lambda(A)$, which finishes the proof.

Alternative argument which does not use the Closed Graph Theorem : A bounded operator defined on a dense subset of a Banach space has a unique bounded extension defined on the whole Banach space. Such an extension is obtained as follows. If $x \in X$, take a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathfrak{D}_\lambda(A)$ for which $x = \lim\limits_{n\to+\infty} u_n$ and define

$$R_\lambda(A)x := \lim_{n\to+\infty} R_\lambda(A)u_n$$

The sequence $(R_\lambda(A)u_n)_{n\in\mathbb{N}}$ is Cauchy, hence converges in X, as was shown above. One easily checks that the definition of $R_\lambda(A)x$ is independent of the chosen sequence $(u_n)_{n\in\mathbb{N}}$. Moreover, by continuity of the norm

$$||R_\lambda(A)x|| = ||\lim_{n\to+\infty} R_\lambda(A)u_n|| = \lim_{n\to+\infty} ||R_\lambda(A)u_n|| \leq \lim_{n\to+\infty} ||R_\lambda(A)|| \cdot ||u_n|| = ||R_\lambda(A)|| \cdot ||x||$$

meaning that the extension $R_\lambda(A)x$ is still bounded with unchanged norm. As $(A-\lambda I)$ is also bounded, one has

$$(A-\lambda I)R_\lambda(A)x = \lim_{n\to+\infty} (A-\lambda I)R_\lambda(A)u_n = \lim_{n\to+\infty} u_n = x$$

for all $x \in X$. Similarly,

$$R_\lambda(A)(A-\lambda I)x = x \quad \forall x \in X$$

which concludes the proof.

5. (a) By assumption, there exists $M \in [0,\infty)$ such that $\sup_{n\geqslant 1}|\theta_n| = M$. Hence,

$$\|Au\|^2 = \sum_{n\geqslant 1} |\theta_n u_n|^2 \leqslant M^2 \sum_{n\geqslant 1} |u_n|^2 = M^2 \|u\|^2, \quad \forall u = (u_n)_{n\geqslant 1} \in \ell^2.$$

This implies that A is bounded, with $\|A\| \leqslant M$.

(b) For each $n_0 \geqslant 1$, θ_{n_0} is an eigenvalue of A, with eigenvector $e_{n_0} = (\delta_{n,n_0})_{n\geqslant 1}$.
Hence, $\sigma_p(A) = (\theta_n)_{n\geqslant 1}$.

Since $\sigma(A)$ is closed, it follows that

$$\overline{(\theta_n)}_{n\geqslant 1} \subset \sigma(A). \tag{1}$$

On the other hand, if $\lambda \notin \overline{(\theta_n)}_{n\geqslant 1}$, for any $v = (v_n)_{n\geqslant 1} \in \ell^2$, the equation

$$(A-\lambda I)u = v \tag{2}$$

can be solved in ℓ^2. Indeed, since $|\theta_n - \lambda|$ is bounded away from zero, $u_n = (\theta_n - \lambda)^{-1}v_n$ yields a unique sequence $u = (u_n)_{n\geqslant 1} \in \ell^2$ satisfying (2). Hence,

$$\lambda \notin \overline{(\theta_n)}_{n\geqslant 1} \implies \lambda \in \rho(A). \tag{3}$$

We conclude from (1) and (3) that $\sigma(A) = \overline{(\theta_n)}_{n\geqslant 1}$.

(c) Consider a limit point $\lambda \in \overline{(\theta_n)}_{n\geqslant 1} \setminus (\theta_n)_{n\geqslant 1}$. Then there exists a subsequence $(\theta_{n_j})_{j\geqslant 1}$ such that $\theta_{n_j} \to \lambda$ as $j \to \infty$. Now consider the sequence $(e_{n_j})_{j\geqslant 1} \subset \ell^2$ defined by $e_{n_j} = (\delta_{n,n_j})_{n\geqslant 1}$, for all $j \geqslant 1$. We have $\|e_{n_j}\| = 1$ for all $j \geqslant 1$ and

$$\left\|(A-\lambda I)^{-1}e_{n_j}\right\|^2 = \sum_{n\geqslant 1} |\theta_n - \lambda|^{-2}\delta_{n,n_j} = |\theta_{n_j} - \lambda|^{-2} \to \infty \quad (j \to \infty).$$

(d) A is symmetric if and only if $(\theta_n)_{n\geqslant 1} \subset \mathbb{R}$.

6. Define T on $C[0,1]$ by

$$Tu(x) = t(x)u(x), \quad x \in [0,1],$$

where $t \in C[0,1]$ is any function whose range is equal to $[a,b]$. Then $(T-\lambda I)u = v$ can be solved uniquely in $C[0,1]$ by the formula

$$u(x) = (t(x)-\lambda)^{-1}v(x), \quad x \in [0,1],$$

if and only if $\lambda \notin [a,b]$. Hence, $\sigma(T) = [a,b]$.