Homework Solution Week 12

1. The spectral family of X is given by

$$E_\lambda\psi = \chi_{(-\infty,\lambda]}u, \quad \lambda\in\mathbb{R},\ \psi\in L^2(\mathbb{R}).$$

Hence, the distribution function of X is

$$F_X(\lambda) = \langle E_\lambda\psi,\psi\rangle = \int_{\mathbb{R}} \chi_{(-\infty,\lambda]}(x)|\psi(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\lambda} |\psi(x)|^2\,\mathrm{d}x$$

and the result follows from relations (4.2.6) of the lecture notes.

2. That $(U_a)_{a\in\mathbb{R}}$ is unitary group is easy to check. To prove strong continuity, dominated convergence cannot be applied directly and we use an approximation by smooth functions.

Let $\psi\in L^2(\mathbb{R})$, $a\in\mathbb{R}$, and consider a sequence $a_n\to a$. Let $\varepsilon>0$ and $\varphi\in C^1(\mathbb{R})$ a function with compact support K such that

$$\|\psi-\varphi\|_{L^2}^2 < \varepsilon.$$

Then there is a constant $C>0$ (independent of φ, ε and n) such that

$$\begin{aligned}\int_{\mathbb{R}} |\psi(x-a_n)-\psi(x-a)|^2\,\mathrm{d}x \leqslant C\Big(&\int_{\mathbb{R}} |\psi(x-a_n)-\varphi(x-a_n)|^2\,\mathrm{d}x\\ &+\int_{\mathbb{R}} |\varphi(x-a_n)-\varphi(x-a)|^2\,\mathrm{d}x\\ &+\int_{\mathbb{R}} |\varphi(x-a)-\psi(x-a)|^2\,\mathrm{d}x\Big).\end{aligned}$$

The first and last integrals in the right-hand side are $<\varepsilon$. On the other hand, by the mean-value theorem, there exists $C_1>0$ such that

$$|\varphi(x-a_n)-\varphi(x-a)|^2 \leqslant C_1|a_n-a|^2.$$

Hence,

$$\int_{\mathbb{R}} |\psi(x-a_n)-\psi(x-a)|^2\,\mathrm{d}x \leqslant 2C\varepsilon + C_1|K||a_n-a|^2 \longrightarrow 2C\varepsilon, \quad n\to\infty.$$

Since $\varepsilon>0$ is arbitrary, we conclude that

$$\int_{\mathbb{R}} |\psi(x-a_n)-\psi(x-a)|^2\,\mathrm{d}x \longrightarrow 0, \quad n\to\infty.$$

3. For any $\psi\in\mathfrak{D}(XP-PX)=\mathfrak{D}_{XP}\cap\mathfrak{D}_{PX}$, there holds

$$\begin{aligned}(XP-PX)\psi(x) = X(P\psi)(x)-P(X\psi)(x) &= x\frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) - \frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}(x\psi(x))\\ &= \frac{\hbar}{i}\Big(x\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)-\psi(x)-x\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)\Big) = -\frac{\hbar}{i}\psi(x) = i\hbar\psi(x).\end{aligned}$$

4. For $\psi\in L^2(\mathbb{R})$, let $S=A-\langle A\rangle_\psi$ and $T=B-\langle B\rangle_\psi$. Then $\mathfrak{D}(ST-TS)=\mathfrak{D}(AB-BA)$ and a direct calculation shows that

$$ST-TS = AB-BA = C.$$

It follows that

$$|\langle C\rangle_\psi| = |\langle (ST-TS)\psi,\psi\rangle| \leqslant |\langle ST\psi,\psi\rangle| + |\langle TS\psi,\psi\rangle| = |\langle T\psi,S\psi\rangle| + |\langle S\psi,T\psi\rangle| \leqslant 2\,\|S\psi\|\,\|T\psi\|,$$

where

$$\|S\psi\| = \|(A-\langle A\rangle_\psi)\psi\| = \Delta_\psi(A)$$

and

$$\|T\psi\| = \|(B - \langle B\rangle_\psi)\psi\| = \Delta_\psi(B).$$