Homework Solution Week 11

1. (a) For all $u \in \mathcal{H}$,

$$\|Uu\|^2 = \langle Uu, Uu\rangle = \langle u, u\rangle = \|u\|^2 . \tag{1}$$

Hence, $\|U\| = 1$.

(b) It follows from () that $Uu = 0$ iff $u = 0$. Hence, $\ker U = \{0\}$ and U is one-to-one. Since U is also surjective, we conclude that U is invertible on $\mathcal{H}$. Now, for $u \in \mathcal{H}$, let $v = U^{-1}u$. Then, for all $w \in \mathcal{H}$,

$$\langle Uw, u\rangle = \langle Uw, Uv\rangle = \langle w, v\rangle = \left\langle w, U^{-1}u\right\rangle .$$

By uniqueness of U^*, it follows that $U^* = U^{-1}$. Furthermore, for all $u, v \in \mathcal{H}$,

$$\left\langle U^{-1}u, U^{-1}v\right\rangle = \left\langle u, (U^{-1})^*U^{-1}v\right\rangle = \left\langle u, (U^*)^{-1}U^{-1}v\right\rangle = \left\langle u, (U^*)^{-1}U^*v\right\rangle = \langle u, v\rangle .$$

Since U^{-1} is surjective, it follows that it is unitary.

We have just seen that $U \in \mathcal{B}(\mathcal{H})$ is unitary $\implies UU^* = U^*U = I$. For the converse, observe that $UU^* = U^*U = I \implies U$ surjective and, for all $u, v \in \mathcal{H}$,

$$\langle Uu, Uv\rangle = \langle u, U^*Uv\rangle = \langle u, v\rangle ,$$

hence U is unitary.

2. Suppose $(U_t)_{t\in\mathbb{R}}$ is a weakly continuous one-parameter unitary group: for all $t_0 \in \mathbb{R}$, for all $u, v \in \mathcal{H}$,

$$\lim_{t\to t_0} \langle U_t u, v\rangle = \langle U_{t_0} u, v\rangle .$$

Then, given $t_0 \in \mathcal{H}$ and $u \in \mathcal{H}$,

$$\begin{aligned}\|U_t u - U_{t_0}u\|^2 &= \langle U_t u - U_{t_0}u, U_t u - U_{t_0}u\rangle = \langle U_t u, U_t u\rangle - 2\mathrm{Re}\,\langle U_t u, U_{t_0}u\rangle + \langle U_{t_0}u, U_{t_0}u\rangle \\ &= \langle u, u\rangle - 2\mathrm{Re}\,\langle U_t u, U_{t_0}u\rangle + \langle u, u\rangle \longrightarrow \langle u, u\rangle - 2\mathrm{Re}\,\langle U_{t_0}u, U_{t_0}u\rangle + \langle u, u\rangle = 0, \quad t \to t_0.\end{aligned}$$