Homework Solution Week 10

1. The proof follows along the same lines as that for bounded symmetric operators.

2. (a) Integrating by parts twice yields

$$\langle Hu, v\rangle = \int_{\mathbb{R}}(-u'')\bar{v} = \int_{\mathbb{R}} u(-\bar{v}'') = \langle u, Hv\rangle\,, \quad \forall u, v \in C_c^\infty(\mathbb{R}).$$

(b) Let

$$\mathfrak{D} := \left\{v \in L^2(\mathbb{R})\,;\, v \in C^1(\mathbb{R}),\ v' \in AC[a,b] \text{ for any } -\infty < a < b < \infty,\ v'' \in L^2(\mathbb{R})\right\}.$$

We first show that $\mathfrak{D} \subseteq \mathfrak{D}_{H^*}$ and $H^*v = -v''$ for all $v \in \mathfrak{D}$. Consider $v \in \mathfrak{D}$. Integrating by parts twice, we have

$$\langle Hu, v\rangle = \int_{\mathbb{R}}(-u'')\bar{v} = \int_{\mathbb{R}} u(-\bar{v}''), \quad \forall u \in \mathfrak{D}_H. \tag{1}$$

Hence,

$$|\langle Hu, v\rangle| \leqslant \|u\|\,\|v''\|\,, \quad \forall u \in \mathfrak{D}_H.$$

It follows that $u \mapsto \langle Hu, v\rangle$ is bounded on $\mathfrak{D}_H$, so $v \in \mathfrak{D}_{H^*}$ and (1) implies that $H^*v = -v''$.

Conversely, suppose that $v \in \mathfrak{D}_{H^*}$. Then $H^*v \in L^2(\mathbb{R}) \subset L^1_{\text{loc}}(\mathbb{R})$. Hence, the function Φ defined by

$$\Phi(x) = \int_0^x \int_0^y H^*v(z)\,\mathrm{d}z\,\mathrm{d}y$$

satisfies

$$\Phi \in C^1(\mathbb{R}),\ \Phi' \in AC[a,b] \text{ for any } a < b,\ \Phi'' = H^*v.$$

Furthermore, since $\langle Hu, v\rangle = \langle u, H^*v\rangle$ for all $u \in \mathfrak{D}_H$, we deduce that

$$-\int_{\mathbb{R}} u''\bar{v} = \int_{\mathbb{R}} u\overline{H^*v} = \int_{\mathbb{R}} u\overline{\Phi''} = \int_{\mathbb{R}} u''\overline{\Phi}, \quad \forall u \in C_c^\infty(\mathbb{R}).$$

By the Du Bois-Reymond Lemma (a variant of the Fundamental Lemma of the Calculus of Variations), there exist $c_0, c_1 \in \mathbb{C}$ such that $v(x) = -\Phi(x) + c_1 x + c_0$. We conclude that, indeed, $v \in \mathfrak{D}$.

(c) We will prove that $\mathrm{rge}(H \pm iI)$ are dense in $L^2(\mathbb{R})$. By Theorem 3.5.6, this implies that H is essentially selfadjoint. To this end, recall that

$$\mathrm{rge}(H \pm iI)^\perp = \ker(H^* \mp iI),$$

so we need only show $\ker(H^* \mp iI) = \{0\}$. Now,

$$u \in \ker(H^* \mp iI) \iff u \in \mathfrak{D}_{H^*} \text{ and } u'' = \mp iu.$$

Solving the differential equations yields two independent solutions

$$u_1^\pm(x) = \exp\left(\frac{(1 \pm i)x}{\sqrt{2}}\right), \quad u_2^\pm(x) = \exp\left(-\frac{(1 \pm i)x}{\sqrt{2}}\right).$$

Since neither of them belongs to $L^2(\mathbb{R})$, we conclude that, indeed, $\ker(H^* \mp iI) = \{0\}$.

3. (a) To see that X is not bounded, observe that, for all $n \geqslant 1$, $u_n := \chi_{[n,n+1)}$ satisfies

$$\|u_n\| = 1 \quad \text{and} \quad \|Xu_n\|^2 = \int_n^{n+1} x^2\,\mathrm{d}x > n^2.$$

We now observe that $\mathfrak{D}_X \subsetneq L^2(\mathbb{R})$ but $\mathfrak{D}_X$ is dense in $L^2(\mathbb{R})$. Indeed, the function $u : \mathbb{R} \to \mathbb{R}$ defined by

$$u(x) = \begin{cases} 1/x & \text{if } x \geqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $L^2(\mathbb{R}) \setminus \mathfrak{D}_X$. Furthermore, $C_c(\mathbb{R}) \subset \mathfrak{D}_X$ and $C_c(\mathbb{R})$ dense in $L^2(\mathbb{R})$ implies $\mathfrak{D}_X$ dense in $L^2(\mathbb{R})$. Hence, X^* exists. To show that $X = X^*$, we proceed in two steps.

First, $X \subseteq X^*$. Indeed, for all $u, v \in \mathfrak{D}_X$,

$$\langle Xu, v\rangle = \int_{\mathbb{R}} xu(x)\overline{v(x)}\,\mathrm{d}x = \int_{\mathbb{R}} u(x)\overline{xv(x)}\,\mathrm{d}x = \langle u, Xv\rangle\,.$$

Next, $\mathfrak{D}_{X^*} \subseteq \mathfrak{D}_X$. Let $v \in \mathfrak{D}_{X^*}$. It follows from the fundamental relation $\langle Xu, v\rangle = \langle u, X^*v\rangle$ that

$$\int_{\mathbb{R}} u(x)(\overline{xv(x) - (X^*v)(x)})\,\mathrm{d}x = 0, \quad \forall u \in \mathfrak{D}_X.$$

Let $a < b$ and choose $u : \mathbb{R} \to \mathbb{C}$ defined by

$$u(x) = \begin{cases} xv(x) - (X^*v)(x) & \text{if } x \in (a,b), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_a^b |xv(x) - (X^*v)(x)|^2\,\mathrm{d}x = 0$$

and it follows that $xv(x) = (X^*v)(x)$ a.e. on (a,b). Since this interval was chosen arbitrarily, we conclude that $xv(x) = (X^*v)(x)$ a.e. on $\mathbb{R}$. Hence, $v \in \mathfrak{D}_X$.

(b) Using the same arguments as for the operator X acting on $L^2[0,1]$ (Weeks 3 and 5), we obtain $\sigma(X) = \sigma_c(X) = \mathbb{R}$ and $E_\lambda u = \chi_{(-\infty,\lambda]}u$ for all $\lambda \in \mathbb{R}$.