Homework Week 9

1. Let $\mathcal{H} = L^2[0,1]$. We continue here our study of the differential operator $i\frac{\mathrm{d}}{\mathrm{d}x}$. We now consider $D_3 : \mathfrak{D}_{D_3} \subset \mathcal{H} \to \mathcal{H}$ defined by

$$D_3 := i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathfrak{D}_{D_3} := C_0^1(0,1).$$

   (a) Show that D_3 is unbounded.

   (b) Prove that the adjoint ${D_3}^*$ of D_3 exists and is given by

$${D_3}^* = i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathfrak{D}_{{D_3}^*} = \mathfrak{D} := \left\{u \in AC[0,1]\,;\, u' \in L^2[0,1]\right\}.$$

   Hint: For any subinterval $[a,b] \subseteq [0,1]$, construct a sequence $(u_n) \subset \mathfrak{D}_{D_3}$ which converges to $\chi_{[a,b]}$. Use this sequence to compute $\int_a^b ({D_3}^*u)(x)\,\mathrm{d}x$ for any $u \in \mathfrak{D}_{{D_3}^*}$.

   (c) Observe the inclusion relations between $\mathfrak{D}_{D_3}, \mathfrak{D}_{{D_3}^*}$ and $\mathfrak{D}_{D_1}, \mathfrak{D}_{{D_1}^*}$.

   (d) Prove that the closure $\overline{D_3}$ of D_3 is given by

$$\overline{D_3} = i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathfrak{D}_{\overline{D_3}} = \left\{u \in AC[0,1]\,;\, u' \in L^2[0,1] \text{ and } u(0) = u(1) = 0\right\}.$$

2. Prove Theorem 3.6.5 from the lecture notes : Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a spectral family and $f : \mathbb{R} \to \mathbb{C}$ be an E-measurable function. Then the following properties hold true.

   (a) $u \in \mathfrak{D}_{E(f)} \iff \|E(f)u\|^2 = \int |f|^2\,\mathrm{d}\mu_{\|E_\lambda u\|^2} < +\infty$.

   (b) If f is bounded, then $\mathfrak{D}_{E(f)} = \mathcal{H}$, $E(f) \in \mathcal{B}(H)$ and $\|E(f)\| \leq \operatorname{ess\,sup}_{\lambda\in\mathbb{R}} |f(\lambda)|$.

   (c) If $f(\lambda) = 1$ for all $\lambda \in \mathbb{R}$, then $E(f) = I$.

   (d) For all $u \in \mathfrak{D}_{E(f)}$,

$$\langle E(f)u, u\rangle = \int f(\lambda)\,\mathrm{d}\mu_{\|E_\lambda u\|^2}$$

   (e) For $a, b \in \mathbb{C}$ and any E-measurable function $g : \mathbb{R} \to \mathbb{C}$,

$$aE(f) + bE(g) \subseteq E(af + bg), \quad \mathfrak{D}_{E(f)+E(g)} = \mathfrak{D}_{E(|f|+|g|)}$$

   (f) For all $\mu \in \mathbb{R}$,

$$E_\mu E(f) \subseteq E(f)E_\mu$$

   with equality if f is bounded.

(g) For any E-measurable functions $f, g : \mathbb{R} \to \mathbb{C}$,

$$E(f)E(g) \subseteq E(fg), \quad \mathfrak{D}_{E(f)E(g)} = \mathfrak{D}_{E(g)} \cap \mathfrak{D}_{E(fg)}$$

Hint: For the equality of domains, show that

$$\int_{-\infty}^{+\infty} |f(\rho)|^2 \, \mathrm{d}\mu_{||E_\rho E(g)u||^2} = \int_{-\infty}^{+\infty} |f(\rho)g(\rho)|^2 \, \mathrm{d}\mu_{||E_\rho u||^2}$$

for any E-measurable functions $f, g : \mathbb{R} \to \mathbb{C}$. Try to prove it first when f is a step function.

(h) $\mathfrak{D}_{E(f)}$ is dense.

Hint: For $u \in \mathcal{H}$, try the sequence $u_n := E(\chi_{[-n,n]} \circ f)u$.

(i)

$$E(\overline{f}) = E(f)^*, \quad \mathfrak{D}_{E(f)^*} = \mathfrak{D}_{E(f)}$$

(j) The operator $E(f)$ is normal, i.e. $E(f)E(f)^* = E(f)^*E(f)$.