Homework Week 8

1. Show that the operators $\mathbf{U}, \mathbf{V} : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ defined by
$$\mathbf{U}(u,v) = (v,u) \quad \text{and} \quad \mathbf{V}(u,v) = (v,-u), \quad (u,v) \in \mathcal{H} \times \mathcal{H},$$
are unitary.

   Recall: A bounded operator A is unitary iff $AA^* = A^*A = I$.

2. (a) Prove that, if a densely defined operator T is closable, then $\left(\overline{T}\right)^* = T^*$.

   (b) Show that a densely defined operator T is closable if and only if T^* is densely defined, in which case $\overline{T} = T^{**}$.

   Hint: First show that $(\mathbf{V}X)^\perp = \mathbf{V}X^\perp$ for any subspace $X \subset \mathcal{H} \times \mathcal{H}$.

3. Let $\mathcal{H} = L^2[0,1]$ and consider the differential operator $D_1 : \mathfrak{D}_{D_1} \subset \mathcal{H} \to \mathcal{H}$ defined by
$$D_1 := i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathfrak{D}_{D_1} := \left\{ u \in AC[0,1]\,;\, u' \in L^2[0,1] \text{ and } u(0) = u(1) = 0 \right\}.$$

   (a) Show that D_1 is unbounded and symmetric.

   (b) Prove that the adjoint ${D_1}^*$ of D_1 is given by
$${D_1}^* = i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathfrak{D}_{{D_1}^*} = \mathfrak{D} := \left\{ u \in AC[0,1]\,;\, u' \in L^2[0,1] \right\}.$$

   Hint: To prove that $\mathfrak{D}_{{D_1}^*} \subseteq \mathfrak{D}$, consider $v \in \mathfrak{D}_{{D_1}^*}$ and use $\langle D_1 u, v \rangle = \langle u, D_1^* v \rangle$, for a well chosen $u \in \mathfrak{D}_{D_1}$ (constructed using v and $D_1^* v$).

   (c) Prove that ${D_1}^{**} = D_1$.

   (d) Now consider $\mathcal{H} = L^2(\mathbb{R})$ and the operator $D_2 : \mathfrak{D}_{D_2} \subset \mathcal{H} \to \mathcal{H}$ defined by $D_2 := i\frac{\mathrm{d}}{\mathrm{d}x}$ and
$$\mathfrak{D}_{D_2} := \left\{ u \in L^2(\mathbb{R})\,;\, u \in AC[a,b] \text{ for any } -\infty < a < b < \infty \text{ and } u' \in L^2(\mathbb{R}) \right\}.$$
   Prove that D_2 is selfadjoint.

   Hint: (i) To show that D_2 is symmetric, prove and use the following lemma:
$$u, u' \in L^2(\mathbb{R}) \implies u(x) \to 0, \ |x| \to \infty.$$
   (ii) To prove $\mathfrak{D}_{{D_2}^*} \subseteq \mathfrak{D}_{D_2}$, extend the arguments in (b) to any closed interval $[a,b]$.

4. Let X be an infinite-dimensional $\mathbb{R}$-normed space and let Y be a $\mathbb{R}$-normed space, $Y \neq \{0\}$. Show that there exists an unbounded linear operator $T : X \to Y$ which is defined on the whole space X.

   Hint: First use Zorn's Lemma to show the existence of a Hamel basis of X, i.e. an uncountable family $\{v_i\}_{i \in I} \subset X$ such that any element $x \in X \setminus \{0\}$ can be written in a unique way as a finite linear combination, $x = \sum_{n=1}^{N} a_n v_{i_n}, \ i_1, \ldots, i_N \in I, N \in \mathbb{N}$.