Homework Week 7

1. Consider an operator $T : \mathfrak{D}_T \subset \mathcal{H} \to \mathcal{H}$, and suppose that T is bounded on $\mathfrak{D}_T$, in the sense that there is a constant $C \geqslant 0$ such that $\|Tu\| \leqslant C\|u\|$, for all $u \in \mathfrak{D}_T$. Show that T can be extended to a bounded linear operator on $\mathcal{H}$. (*Remark:* It is not assumed that $\mathfrak{D}_T$ is dense).

2. Let $S : \mathfrak{D}_S \subset \mathcal{H} \to \mathcal{H}$ be a one-to-one operator. Consider the following properties.
   (i) S is closed.
   (ii) $\operatorname{rge} S$ is dense.
   (iii) $\operatorname{rge} S$ is closed.
   (iv) There is a constant C such that $\|Su\| \geqslant C\|u\|$, for all $u \in \mathfrak{D}_S$.
   (a) Prove that (i)-(ii)-(iii) imply (iv). *Hint:* Apply the Closed Graph Theorem to S^{-1}.
   (b) Prove that (ii)-(iii)-(iv) imply (i).
   (c) Prove that (i) and (iv) imply (iii).

3. Prove that, if T^{-1}, T^* and $(T^{-1})^*$ exist, then $(T^*)^{-1}$ also exists, and $(T^{-1})^* = (T^*)^{-1}$.

4. We define an operator T on $L^2(\mathbb{R})$ by $(Tu)(x) = \phi(x)u(x)$, where $\phi \in L^\infty(\mathbb{R})$.
   (a) Show that T is bounded and compute its norm.
   (b) Find T^*. Under what condition is $T = T^*$?
   (c) If S is defined by $(Su)(x) = \psi(x)u(x)$ with $\psi \in L^\infty(\mathbb{R})$, find TS and $(TS)^*$.

5. Consider again the multiplication operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(Tu)(x) = \phi(x)u(x)$. Suppose now that $\phi : \mathbb{R} \to \mathbb{C}$ is continuous and satisfies $\lim_{x\to+\infty} |\phi(x)| = +\infty$.
   (a) Find the domain of T and show that T is unbounded.
   (b) Find T^*.
   Hint: Start by showing that $\mathfrak{D}_T \subseteq \mathfrak{D}_{T^*}$ and $T^*v = \bar{\phi}v$, for all $v \in \mathfrak{D}_T$.