Homework Week 6

The problems with a $\star$ are optional.
Solutions to problems 3, 4 and 5 can be found in Kreyszig's book.

1. From Theorems 2.3.1 and 2.3.2, deduce the structure of the spectral family and the spectral decomposition of
   (a) a symmetric matrix;
   (b) a compact symmetric operator with infinitely many eigenvalues.

2. Verify that the spectral family of the operator $X : L^2[0,1] \to L^2[0,1]$ studied last week satisfies the conclusions of Theorems 2.3.1 and 2.3.2.

$\star$ 3. The goal of this problem is to prove Theorem 2.3.1. First, the whole proof can be reduced to checking (2.3.1). Explain why. Then, to prove (2.3.1), proceed as follows.
   (a) To prove that $\ker(S - \lambda_0 I) \supset \mathrm{rge}(E_{\lambda_0+0} - E_{\lambda_0})$, use (2.2.4).
   (b) Show that, if $u \in \ker(S - \lambda_0 I)$ then $F_0 u = u$, where $F_0 := E_{\lambda_0+0} - E_{\lambda_0}$ — explain why this is enough. To do this, use Corollary 2.2.3 with $p(\lambda) = (\lambda - \lambda_0)^2$ to prove that

$$\langle E_{\lambda_0-\epsilon} u, u\rangle = \langle u - E_{\lambda_0+\epsilon} u, u\rangle = 0, \quad \forall \epsilon > 0.$$

$\star$ 4. This problem is devoted to the proof of Theorem 2.3.2. We suggest to use the following characterization of $\rho(S)$: $\lambda_0 \in \rho(S)$ if and only if there exists $\gamma > 0$ such that

$$\|(S - \lambda_0 I)u\| \geqslant \gamma \|u\|, \quad u \in \mathcal{H}. \tag{1}$$

   (a) To prove that the constancy condition implies $\lambda_0 \in \rho(S)$, use Corollary 2.2.3 with $p(\lambda) = (\lambda - \lambda_0)^2$, and (1).
   (b) To show that λ_0 is a point of constancy if it is in the resolvent set, proceed by contradiction using again (1) and Corollary 2.2.3..

   Hint: The identities $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda,\ \lambda \leqslant \mu$, can be useful here.

5. Let S be a symmetric operator, (E_λ) the corresponding spectral family with lower and upper bounds $m, M \in \mathbb{R}$, and $f \in C^0([m,M],\mathbb{R})$.
   (a) Prove that there exists a unique operator $f(S) \in \mathcal{B}(\mathcal{H})$ such that any sequence of real polynomials (p_n) converging uniformly to f on $[m,M]$ satisfies

$$\|p_n(S) - f(S)\| \longrightarrow 0, \quad n \to \infty.$$

   Hint: First show that $\|p(S)\| \leqslant \sup_{[m,M]} |p|$ for any real polynomial p.

   (b) Prove that

$$f(S) = \int_m^{M+\varepsilon} f(\lambda)\, \mathrm{d}E_\lambda$$

   and

$$\langle f(S)u, v\rangle = \int_m^{M+\varepsilon} f(\lambda)\, \mathrm{d}\langle E_\lambda u, v\rangle, \quad \forall u, v \in \mathcal{H}.$$