Homework Week 5

1. Suppose $\mathcal{H} \neq \{0\}$.

   (a) Find the spectral family and verify the conclusions of Spectral Theorem I for the zero operator $S : \mathcal{H} \to \mathcal{H},\ Su = 0$ for all $u \in \mathcal{H}$.

   (b) Find the spectral family and verify the conclusions of Spectral Theorem I for the identity operator $I : \mathcal{H} \to \mathcal{H}$.

2. Consider the operator $X : L^2[0,1] \to L^2[0,1]$, $(Xu)(x) = xu(x)$. Prove that the spectral family $(E_\lambda)_{\lambda\in\mathbb{R}}$ of X is given by

$$E_\lambda u = \begin{cases} 0 & \text{if} \quad \lambda \leqslant 0, \\ \chi_{[0,\lambda]} u & \text{if} \quad \lambda \in (0,1], \\ u & \text{if} \quad \lambda > 1, \end{cases}$$

   where $\chi_{[0,\lambda]}$ is the characteristic function of the interval $[0, \lambda]$.

   Hint: Start by showing that $|X - \lambda I|$ is the operator $T_\lambda : L^2[0,1] \to L^2[0,1]$ defined by $(T_\lambda u)(x) = |x-\lambda| u(x),\ x \in [0,1]$. Then find the corresponding projection $E_+(\lambda)$ appearing in the proof of Spectral Theorem I.

3. Let $S \in \mathcal{B}(\mathcal{H})$ be a symmetric operator and $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a corresponding spectral family given by Spectral Theorem I. Prove that, for any real polynomial p,

$$\langle p(S)u, v\rangle = \int_m^{M+\varepsilon} p(\lambda)\,\mathrm{d}\langle E_\lambda u, v\rangle, \quad \forall u, v \in \mathcal{H}. \tag{1}$$

   Hint: Start by proving the case $v = u$, using Theorem A.2.1 and Lemma 2.2.2.

4. Prove the uniqueness of the spectral family given by Spectral Theorem I.

   Hint: Given two spectral families $(E_\lambda)_{\lambda\in\mathbb{R}}, (F_\lambda)_{\lambda\in\mathbb{R}}$ and a fixed $u \in \mathcal{H}$, introduce the function $\phi(\lambda) := \langle E_\lambda u, u\rangle - \langle F_\lambda u, u\rangle$. Then use (1), Theorem A.3.3 and Theorem A.3.4.