Homework Week 4

1. Let $A \in \mathcal{B}(\mathcal{H})$. Prove that $r_\sigma(A) \leqslant \|A\|$.

   Hint: Use the contraction mapping principle.

2. An operator $A \in \mathcal{B}(\mathcal{H})$ is called **normal** if $AA^* = A^*A$. Prove that $r_\sigma(A) = \|A\|$ for any normal operator $A \in \mathcal{B}(\mathcal{H})$.

   Hint: Consider the symmetric operator $T = AA^*$ and use the Spectral Radius Theorem to compute $r_\sigma(T)$ in two different ways.

3. Show that the relation $A \geqslant B$ defines a partial order on $\mathcal{B}(\mathcal{H})$.

   Recall: A partial order is a binary relation which is reflexive, transitive and antisymmetric.

4. Let $A : \ell^2 \to \ell^2$ be the **double right shift**, defined by $(u_1, u_2, \dots) \mapsto (0, 0, u_1, u_2, \dots)$.
   (a) Show that A is bounded and compute $\|A\|$.
   (b) Find A^*.
   (c) Find $\sigma_p(A), \sigma_r(A)$ and $\sigma_c(A)$.
   (d) Is A positive? Find $B : \ell^2 \to \ell^2$ such that $A = B^2$.

5. Let $a \in L^\infty[0,1]$ and consider the **multiplication operator** $A : L^2[0,1] \to L^2[0,1]$ defined by $(Au)(x) = a(x)u(x)$.
   (a) Show that A is bounded and compute $\|A\|$.
   (b) Find a condition on a for A to be positive and, in this case, find the square root of A.
   (c) Show that:
      (i) $\sigma_p(A) = \big\{\lambda \in \mathbb{C}\,;\, \big|\{x \in [0,1]\,;\, a(x) = \lambda\}\big| > 0\big\}$;
      (ii) $\sigma_r(A) = \emptyset$;
      (iii) $\sigma(A) = \mathrm{essrge}(a)$.

   Recall: $|E|$ is the Lebesgue measure of a Borel set $E \subset [0,1]$, and $\mathrm{essrge}(a)$ is the essential range of a, defined as

$$\mathrm{essrge}(a) := \big\{\lambda \in \mathbb{C}\,;\, \text{for all } \varepsilon > 0 : \big|\{x \in [0,1]\,;\, |a(x) - \lambda| < \varepsilon\}\big| > 0\big\}.$$