Homework Week 3

1. Let $S : \mathcal{H} \to \mathcal{H}$ be bounded and symmetric.
   (a) Prove that all the eigenvalues (if any) of S are real. Show that eigenvectors of S corresponding to distinct eigenvalues are orthogonal.
   (b1) Prove that $\lambda \in \rho(S)$ iff there exists $C > 0$ such that
   $$\|(S - \lambda I)u\| \geqslant C\,\|u\|\,, \quad \forall u \in \mathcal{H}.$$
   (b2) Prove that $\lambda \in \sigma(S)$ iff there exists a sequence $(u_n) \subset \mathcal{H}$ such that
   $$\|u_n\| = 1 \ \forall n \geqslant 1 \quad \text{and} \quad \|(S - \lambda I)u_n\| \to 0 \ (n \to \infty).$$
   (c) Prove that $\sigma(S) \subset \mathbb{R}$ using (b1).
   (d1) Let m and M be the lower bound and upper bound of S, respectively.
   Prove that $\lambda \notin [m, M] \Rightarrow \lambda \in \rho(S)$.
   (d2) Prove that $m, M \in \sigma(S)$.
   Hint: Start by showing that $M \in \sigma(S)$ in case $S \geqslant 0$.

2. Consider a sequence $(S_n)_{n\geqslant 1}$ of symmetric operators, such that $S_n \to S$ strongly in $\mathcal{B}(\mathcal{H})$ as $n \to \infty$. Show that S is symmetric.
   Hint: Apply the triangle inequality to $\|S - S^*\|$.

3. Find a Hilbert space $\mathcal{H}$ and a sequence of operators $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ which converges strongly but not in operator norm.

4. Let $S, B \in \mathcal{B}(\mathcal{H})$.
   (a) If S is symmetric, show that $SB = BS \Rightarrow SB^* = B^*S$.
   (b) If S and B are symmetric, show that SB is symmetric $\Leftrightarrow SB = BS$.
   (c) If S is symmetric, show that any polynomial in S with real coefficients is symmetric.

5. Show that the multiplication operator $X : L^2[0,1] \to L^2[0,1]$ defined by
   $$(Xu)(x) = xu(x), \quad x \in [0,1],$$
   is a bounded symmetric operator without eigenvalues. Find the spectrum of X.