Homework Week 2

1. Let M and N be closed subspaces of the Hilbert space $\mathcal{H}$. Denote by P and Q the associated projections. Prove the following statements. (Drawing a picture in $\mathbb{R}^3$ can help.)

   (a) For all $u, v \in \mathcal{H}$, $\langle Pu, v\rangle = \langle u, Pv\rangle$.

   (b) $M = \operatorname{Im} P$ and $M^\perp = \ker P$.

   (c) $I - P$ is the projection onto $M^\perp$.

   (d) $M \subseteq N$ if and only if $PQ = QP = P$.

   (e) $PQ = 0 \Leftrightarrow QP = 0 \Leftrightarrow M \perp N$.

   (f) For all $u \in \mathcal{H}$, $\langle Pu, u\rangle \leqslant \langle Qu, u\rangle$ if and only if $M \subseteq N$.

2. Prove that a bounded operator $P : \mathcal{H} \to \mathcal{H}$ is a projection if and only if it is symmetric and idempotent (i.e. $P^2 = P$).

3. Let $\mathcal{H}$ be a complex Hilbert space. For $T \in \mathcal{B}(\mathcal{H})$, prove that

   $$\langle Tu, u\rangle = 0 \ \forall u \in \mathcal{H} \implies T = 0.$$

   Find a counterexample in the real case.

   Hint: Write $u = v + \lambda w$ and use special values of $\lambda \in \mathbb{C}$.

4. Prove that, for any bounded operator $A : \mathcal{H} \to \mathcal{H}$,

   $$\mathcal{H} = \ker A^* \oplus \overline{\operatorname{rge} A}.$$

   Hint: (Prove and) use the fact that $\mathcal{X}^{\perp\perp} = \overline{\mathcal{X}}$ for any subspace $\mathcal{X} \subset \mathcal{H}$.

5. We define the **left shift** $S : \ell^2 \to \ell^2$ and the **right shift** $T : \ell^2 \to \ell^2$ by

   $$(Su)_n = u_{n+1}, \ n \geqslant 1$$

   and

   $$(Tu)_1 = 0, \ (Tu)_n = u_{n-1}.$$

   (a) Show that S and T are bounded. Compute $\|S\|$ and $\|T\|$.

   (b) Find S^* and T^*.

   (c) Find $\sigma_p(S), \sigma_p(T), \sigma_r(S), \sigma_r(T), \sigma_c(S)$ and $\sigma_c(T)$.

6. Suppose $S \in \mathcal{B}(\mathcal{H})$ is symmetric. Prove that

   $$\operatorname{Re}\langle Su, v\rangle = \frac{1}{4}\big(\langle S(u+v), u+v\rangle - \langle S(u-v), u-v\rangle\big), \quad \forall u, v \in \mathcal{H}$$

   and

   $$\operatorname{Im}\langle Su, v\rangle = \frac{1}{4}\big(\langle S(u+iv), u+iv\rangle - \langle S(u-iv), u-iv\rangle\big), \quad \forall u, v \in \mathcal{H}.$$

   Deduce the classical polarization identity for $\langle u, v\rangle$.