Homework Week 1

1. Prove that the inner product of a Hilbert space is continuous in each variable, and that the norm is a continuous function.

2. Prove that the norm $\|A\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)}$ of an operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ can be characterized by
$$\|A\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} = \sup_{\|u\|_{\mathcal{H}_1}=1} \|Au\|_{\mathcal{H}_2} = \sup_{\|u\|_{\mathcal{H}_1}\leqslant 1} \|Au\|_{\mathcal{H}_2}.$$

3. Prove the **Closed Graph Theorem**:

   Let $\mathcal{X}_1, \mathcal{X}_2$ be Banach spaces. Then $A : \mathcal{X}_1 \to \mathcal{X}_2$ is bounded if and only if its graph $\boldsymbol{G}_A$ is a closed subset of $\mathcal{X}_1 \times \mathcal{X}_2$.

   Hint: To prove that A is bounded if $\boldsymbol{G}_A$ is closed, use the projections $\pi_j : \boldsymbol{G}_A \to \mathcal{X}_j$ $(j = 1, 2)$ defined by $\pi_1(u, Au) = u,\ \pi_2(u, Au) = Au$, and the bounded inverse theorem.

4. Let $A \in \mathcal{B}(\mathcal{X})$ and $\lambda \in \mathbb{C}$. Show that, if $\mathcal{X}$ is a Banach space and $\lambda \in \rho(A)$, then $(A-\lambda I)^{-1}$ is defined on the whole space $\mathcal{X}$.

   Hint: Use the Closed Graph Theorem.

5. Let $(\theta_n)_{n\geqslant 1} \subset \mathbb{C}$ be a bounded sequence. Consider the **multiplication operator** $A : \ell^2 \to \ell^2$ defined by
$$(Au)_n = \theta_n u_n, \quad \forall n \geqslant 1.$$

   Recall: ℓ^2 is the Hilbert space of complex sequences $(u_1, u_2, \dots)$ such that $\sum_{n\geqslant 1} |u_n|^2 < \infty$, endowed with the inner product $\langle u, v\rangle = \sum_{n\geqslant 1} u_n \overline{v_n}$.

   (a) Prove that A is bounded.

   (b) Find $\sigma_p(A)$ and $\sigma(A)$.

   Hint: Use the fact that $\sigma(A)$ is closed.

   (c) Prove that, if $\lambda \in \sigma(A) \setminus \sigma_p(A)$, then $(A - \lambda I)^{-1}$ is not bounded.

   (d) Determine under which condition on $(\theta_n)_{n\geqslant 1}$ the operator A is symmetric.

6. Let $a, b \in \mathbb{R},\ a < b$. Find an operator $T : C[0, 1] \to C[0, 1]$ such that $\sigma(T) = [a, b]$.