Homework Week 12

1. The position operator X of the quantum particle on the line is defined by
$$\mathfrak{D}_X = \{\psi \in L^2(\mathbb{R})\,;\, x\psi(x) \in L^2(\mathbb{R})\}, \quad (X\psi)(x) = x\psi(x)$$
(cf. Problem 3, Week 10). Using the probabilistic interpretation of the distribution function
$$F_X(\lambda) = \langle E_\lambda \psi, \psi \rangle\,,$$
recover the statement that the probability of finding the particle in a normalized state ψ in the interval $\Delta = [a, b]$ is given by $\int_a^b |\psi(x)|^2 \,\mathrm{d}x$.

2. The group of translations acting on $L^2(\mathbb{R})$ is defined by
$$(U_a\psi)(x) = \psi(x - a), \quad a \in \mathbb{R}.$$
Show that $(U_a)_{a\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group.

3. The momentum operator P of the quantum particle on the line is defined by
$$\mathfrak{D}_P = \{\psi \in L^2(\mathbb{R})\,;\, \psi \text{ is } AC \text{ and } \psi'(x) \in L^2(\mathbb{R})\}, \quad (P\psi)(x) = \frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)$$
(cf. Problem 3, Week 8). Prove that
$$XP - PX = i\hbar I,$$
where I is the identity operator on the domain $\mathfrak{D}_{XP-PX} = \mathfrak{D}_{XP} \cap \mathfrak{D}_{PX}$.

4. Let A and B be selfadjoint operators acting in the Hilbert space $L^2(\mathbb{R})$. Prove that the commutator $C = AB - BA$ satisfies
$$|\langle C\rangle_\psi| \leqslant 2\Delta_\psi(A)\Delta_\psi(B),$$
for all $\psi \in \mathfrak{D}_C = \mathfrak{D}_{AB} \cap \mathfrak{D}_{BA}$.
Hint: First show that $C = ST - TS$ where $S = A - \langle A\rangle_\psi$ and $T = B - \langle B\rangle_\psi$.