Homework Week 11

1. Taking the definition that $U : \mathcal{H} \to \mathcal{H}$ is unitary if U is surjective and $\langle Uu, Uv\rangle = \langle u, v\rangle$ for all $u, v \in \mathcal{H}$, show that a unitary operator $U : \mathcal{H} \to \mathcal{H}$ satisfies the following properties:
   (a) $\|U\| = 1$;
   (b) U is invertible with $U^{-1} = U^*$ unitary.

   Then show that, in fact, $U \in \mathcal{B}(\mathcal{H})$ is unitary if and only if $UU^* = U^*U = I$.

2. Show that a weakly continuous one-parameter unitary group is strongly continuous.

3. Prove Stone's Theorem: To any strongly continuous one-parameter unitary group $(U_t)_{t\in\mathbb{R}}$ corresponds a unique selfadjoint operator A such that

$$U_t = \mathrm{e}^{itA} \quad \text{for all } t \in \mathbb{R}.$$

   Furthermore, $U_t \smile A$ for all $t \in \mathbb{R}$.

   Strategy: Let $A := -iG$ where G is the infinitesimal generator of $(U_t)_{t\in\mathbb{R}}$, and consider the set $\mathfrak{D}$ of all finite linear combinations of elements of the form

$$u_\phi = \int_{\mathbb{R}} \phi(t) U_t u \,\mathrm{d}t,$$

   for $u \in \mathcal{H}$ and $\phi \in C_0^\infty(\mathbb{R})$ (see Appendix C for the meaning of this integral). The theorem is then proved in three steps.

   (i) Using the results of Section C.2, show that $\mathfrak{D}$ is dense in $\mathcal{H}$, and contained in $\mathfrak{D}_A$.

   (ii) Show that A is essentially selfadjoint.

   Hint: Letting $f(t) = \langle U_t u, v\rangle$, note that

$$\langle u, G^* v\rangle = \langle Gu, v\rangle = f'(0), \quad \forall u \in \mathfrak{D}_G, v \in \mathfrak{D}_{G^*}.$$

   (iii) By differentiating $\|(U_t - \mathrm{e}^{it\bar{A}})u\|^2$ with respect to t, show that $U_t = \mathrm{e}^{it\bar{A}}$. Conclude.