Homework Week 10

1. Let $A : \mathfrak{D}_A \subseteq \mathcal{H} \to \mathcal{H}$ be a selfadjoint operator. Prove that $\sigma_r(A) = \emptyset$.

2. Let $\mathcal{H} = L^2(\mathbb{R})$ and define $H : \mathfrak{D}_H \subset \mathcal{H} \to \mathcal{H}$ by
$$H := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathfrak{D}_H := C_c^\infty(\mathbb{R}).$$
   (a) Prove that H is symmetric.

   (b) Prove that H^* is given by $H^* = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $\mathfrak{D}_{H^*} = \mathfrak{D}$, where
$$\mathfrak{D} := \left\{ v \in L^2(\mathbb{R})\,;\, v \in C^1(\mathbb{R}),\ v' \in AC[a,b] \text{ for any } -\infty < a < b < \infty,\ v'' \in L^2(\mathbb{R}) \right\}.$$
   (c) Using Theorem 3.5.6, determine whether H is essentially selfadjoint.

3. Consider the multiplication operator in $\mathcal{H} = L^2(\mathbb{R})$ defined by
$$(Xu)(x) := xu(x), \quad \mathfrak{D}_X := \left\{ u \in L^2(\mathbb{R})\,;\, xu(x) \in L^2(\mathbb{R}) \right\}.$$
   (a) Show that X is unbounded and selfadjoint.

   (b) Find the spectrum and the spectral family of X.