Spectral Theory Oral Exam
List of Questions

Q1: Define the spectrum of a bounded operator A mapping a normed space $\mathcal{X}$ into itself.

Describe the decomposition of the spectrum into point spectrum, continuous spectrum and residual spectrum.

If $\mathcal{X}$ is a Banach space and λ is not in the spectrum of A, what can be said of the operator $(A - \lambda I)^{-1}$?

Q2: Define the adjoint A^* of a bounded operator A acting in a Hilbert space $\mathcal{H}$. Define what it means for A to be symmetric.

How do these notions extend to unbounded operators?

Next, consider a sequence of bounded symmetric operators (A_n) such that $A_n \to A$ in $\mathcal{B}(\mathcal{H})$. Prove that A is also symmetric.

Q3: Show that, for any subspace $\mathcal{X}$ of a Hilbert space $\mathcal{H}$, $\mathcal{X}^{\perp\perp} = \overline{\mathcal{X}}$.

Use this fact to prove that, for any bounded operator $A : \mathcal{H} \to \mathcal{H}$, there holds

$$\mathcal{H} = \ker A^* \oplus \overline{\operatorname{rge} A}.$$

Q4: Define the spectrum of a bounded operator A mapping a normed space $\mathcal{X}$ into itself.

Describe the decomposition of the spectrum into point spectrum, continuous spectrum and residual spectrum.

Prove that the residual spectrum of a bounded symmetric operator is always empty.

Give an example of a non-symmetric operator having an empty residual spectrum.

Q5: Define the adjoint A^* of a bounded operator A acting in a Hilbert space $\mathcal{H}$. Define what it means for A to be symmetric.

Prove that an operator $A \in \mathcal{B}(\mathcal{H})$ is symmetric if and only if its numerical range is real.

If $\mathcal{H}$ is complex, prove that an operator $T \in \mathcal{B}(\mathcal{H})$ satisfies $\langle Tu, u \rangle = 0$ for all $u \in \mathcal{H}$ if and only if $T = 0$.

What can be said if $\mathcal{H}$ is real?

Q6: Give the definition of an orthogonal projection P acting in a Hilbert space $\mathcal{H}$.

What is the numerical range of an orthogonal projection? Justify your answer.

Show that a bounded operator $P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection if and only if P is symmetric and $P^2 = P$.

Q7: Define what is a positive operator acting in a Hilbert space $\mathcal{H}$.

Show that an orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ is always positive.

Now consider two orthogonal projections $P, Q : \mathcal{H} \to \mathcal{H}$. Show that $P \leqslant Q$ if and only if $\operatorname{rge} P \subseteq \operatorname{rge} Q$.

Q8: Show that the multiplication operator $X : L^2[0,1] \to L^2[0,1]$ defined by

$$(Xu)(x) = xu(x), \quad x \in [0,1],$$

is a bounded symmetric operator without eigenvalues. What is the spectrum of X?

Now extend the discussion to the operator $\widetilde{X}$ acting in $L^2(\mathbb{R})$ as

$$(\widetilde{X}u)(x) = xu(x), \quad x \in \mathbb{R}.$$

Q9: Let $(\theta_n)_{n\geqslant 1} \subset \mathbb{C}$ be a bounded sequence. Consider the multiplication operator $A : \ell^2 \to \ell^2$ defined by

$$(Au)_n = \theta_n u_n, \quad \forall n \geqslant 1.$$

Show that A is bounded and find its spectrum. Prove that, if $\lambda \in \sigma(A) \setminus \sigma_p(A)$, then $(A - \lambda I)^{-1}$ is not bounded.

Q10: We define the left shift $S : \ell^2 \to \ell^2$ and the right shift $T : \ell^2 \to \ell^2$ by

$$(Su)_n = u_{n+1},\ n \geqslant 1,$$

and

$$(Tu)_1 = 0,\ (Tu)_n = u_{n-1}.$$

Find S^*, T^*. Then determine $\sigma_p(S), \sigma_p(T), \sigma_r(S), \sigma_r(T), \sigma_c(S)$ and $\sigma_c(T)$.

Q11: Define what is a spectral family, distinguishing the two definitions we gave in the course.

State the spectral theorem for bounded symmetric operators, and explain how the spectral family is constructed in our proof of this theorem.

Find the spectral family of the multiplication operator $X : L^2[0,1] \to L^2[0,1]$ defined by

$$(Xu)(x) = xu(x), \quad x \in [0,1].$$

Q12: Explain the construction of the integral of a function $f \in C^0([m,M])$ with respect to a spectral family with lower bound m and upper bound M.

If S is a bounded symmetric operator with lower bound m and upper bound M, define $f(S)$ using the spectral theorem.

Then show that

$$\langle f(S)u, u\rangle = \int_m^{M+\varepsilon} f(\lambda)\, \mathrm{d}\langle E_\lambda u, u\rangle \quad \forall u \in \mathcal{H}.$$

Q13: Given a bounded symmetric operator S acting in a Hilbert space $\mathcal{H}$, characterize $\sigma(S)$ and $\sigma_p(S)$ using the spectral family of S.

Now construct the spectral family of the multiplication operator $X : L^2[0,1] \to L^2[0,1]$ defined by

$$(Xu)(x) = xu(x), \quad x \in [0,1].$$

Relate your result to the previous question.

Q14: Define the adjoint A^* of an operator $A : \mathfrak{D}_A \subseteq \mathcal{H} \to \mathcal{H}$, where $\mathfrak{D}_A$ is a dense subspace of the Hilbert space $\mathcal{H}$.

Prove that A^* is closed.

Q15: Consider the multiplication operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(Tu)(x) = \phi(x)u(x)$. Suppose that $\phi : \mathbb{R} \to \mathbb{C}$ is continuous and satisfies

$$\lim_{x \to +\infty} |\phi(x)| = +\infty.$$

Find the domain of T and show that T is unbounded. Find T^*.

Q16: Define what it means for an operator T acting in a Hilbert space $\mathcal{H}$ to be closable.

Prove that a densely defined operator T is closable if and only if T^* is densely defined, in which case $\overline{T} = T^{**}$.

Q17: Define what it means for an operator T acting in a Hilbert space $\mathcal{H}$ to be closable.

Define what it means for T to be symmetric.

Show that a symmetric operator T is always closable and find its closure.

Q18: Let $\mathcal{H} = L^2[0,1]$ and consider the differential operator $P : \mathfrak{D}_P \subset \mathcal{H} \to \mathcal{H}$ defined by

$$P = i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathfrak{D}_P = C_0^1(0,1).$$

Show that P is unbounded.

Prove that $\mathfrak{D}_{P^*} = \{u \in AC[0,1]\,;\, u' \in L^2[0,1]\}$ and $P^*v = iv'$ for all $v \in \mathfrak{D}_{P^*}$.

(You can use without proof the Du Bois–Reymond Lemma.)

Q19: Let $\mathcal{H} = L^2(\mathbb{R})$ and consider the differential operator $H : \mathfrak{D}_H \subset \mathcal{H} \to \mathcal{H}$ defined by $H = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$, with domain $\mathfrak{D}_H = C_0^\infty(\mathbb{R})$, the space of C^∞ functions with compact support.

Show that H is symmetric.

Prove that

$$\mathfrak{D}_{H^*} = \{v \in L^2(\mathbb{R})\,;\, v \in C^1(\mathbb{R}),\ v' \in AC[a,b] \text{ for any } -\infty < a < b < \infty,\ v'' \in L^2(\mathbb{R})\}$$

and $H^*v = -v''$ for all $v \in \mathfrak{D}_{H^*}$.

(You can use without proof the Du Bois–Reymond Lemma.)

Q20: Given a spectral family $(E_\lambda)_{\lambda\in\mathbb{R}}$ in a Hilbert space $\mathcal{H}$, explain what is an E-measurable function f. For such an f, construct the operator

$$\begin{aligned} E(f) : \mathfrak{D}_{E(f)} &\longrightarrow \mathcal{H} \\ u &\longmapsto \int f(\lambda)\,\mathrm{d}E_\lambda u. \end{aligned}$$

Show that, for all $u \in \mathfrak{D}_{E(f)}$, $\|E(f)u\|^2 = \int |f|^2 \,\mathrm{d}\mu_{\|E_\lambda u\|^2}$.

Q21: Given a spectral family $(E_\lambda)_{\lambda\in\mathbb{R}}$ in a Hilbert space $\mathcal{H}$, explain what is an E-measurable function f. For such an f, construct the operator

$$\begin{aligned} E(f) : \mathfrak{D}_{E(f)} &\longrightarrow \mathcal{H} \\ u &\longmapsto \int f(\lambda)\,\mathrm{d}E_\lambda u. \end{aligned}$$

Show that, if f is bounded, then $\mathfrak{D}_{E(f)} = \mathcal{H}$. Give an upper bound on $\|E(f)\|$ in this case.

Q22: State the spectral theorem for general selfadjoint operators.

Given a selfadjoint operator A acting in a Hilbert space $\mathcal{H}$, use the spectral theorem to define $U_t = \mathrm{e}^{itA}$ for $t \in \mathbb{R}$. Describe the main properties of this object. In particular prove that, if $t_n \to t^*$, then $U_{t_n}u \to U_{t^*}u$ for all $u \in \mathcal{H}$.

Q23: Define what is a strongly continuous one-parameter unitary group.

State Stone's theorem.

Explain how the Hamiltonian of a quantum system invariant under time translation can be defined using this theorem.

Deduce the equation describing the time evolution of the system. Show that the mean value of the energy is conserved by the evolution.

Q24: Define what it means for a one-parameter unitary group to be: strongly continuous; weakly continuous.

Show that a weakly continuous one-parameter unitary group is strongly continuous.

Show that the operators defined on $L^2(\mathbb{R})$ by

$$(U_a\psi)(x) = \psi(x-a), \quad a \in \mathbb{R},$$

form a strongly continuous one-parameter unitary group.

Use Stone's theorem to derive the definition of the momentum operator for the quantum particle on the real line.

Q25: Define the position operator X and the momentum operator P for the quantum particle on the real line.

State and prove the Heisenberg commutation relation.

State and prove the Heisenberg uncertainty principle.