Fundamental Lemma of Calculus of Variations/Du Bois-Reymond Lemma

Lemma 1. *Let $n \in \mathbb{N}_{\geq 0}$. Let $f \in L^1_{loc}((a,b))$, $-\infty \leq a < b \leq +\infty$, satisfy*

$$\int_a^b f(x)\phi^{(n)}(x)dx = 0 \quad \forall \phi \in C_c^\infty((a,b))$$

Then $f(x)$ is a.e. a polynomial of degree at most $n-1$ (where 0 is what we consider a polynomial of degree -1).

Proof. First, we state a useful observation : Assume that $f \in L^1_{loc}((a,b))$ and for any compact interval $I \subset (a,b)$ which is not a singleton, there exists a polynomial $p_I(x)$ for which $f(x) = p_I(x)$ almost everywhere on I. Then the polynomial representation of f does not depend on I, i.e. $f(x) = p_I(x) = p(x)$ a.e. on (a,b). *Proof :* It suffices to observe that the polynomial $p_I(x)$ is independent of I. If $I, J \subset (a,b)$ are two compact intervals, take a larger compact interval $I, J \subset K \subset (a,b)$ so that $f(x) = p_I(x) = p_K(x)$ on I and $f(x) = p_J(x) = p_K(x)$ on J. This implies $p_I(x) = p_J(x) = p_K(x)$ are the same polynomials.

Thanks to this observation, assume without loss of generality that $f \in L^1([a,b])$. Let $\phi \in C_c^\infty(\mathbb{R})$, $\phi \geq 0$, $||\phi||_{L^1(\mathbb{R})} = 1$ and $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(\varepsilon^{-1}x)$ for $\varepsilon > 0$ be a standard family of mollifiers.

Then $\rho_\varepsilon * f(x)$ converges to $f(x)$ in $L^1([a,b])$ as $\varepsilon \to 0$. In particular, we have pointwise convergence almost everywhere along some subsequence. Moreover, $\rho_\varepsilon * f(x) \in C^\infty(\mathbb{R})$ and $(\rho_\varepsilon * f)^{(n)}(x) = (\rho_\varepsilon^{(n)} * f)(x)$.

Observe now that for $x \in (a,b)$ fixed, $y \mapsto \rho_\varepsilon(x-y)$ has support in $(x - \varepsilon b, x - \varepsilon a) \subset (a,b)$ if $0 < \varepsilon < \min\{(x-a)b^{-1}, (b-x)a^{-1}\}$, where $0^{-1} = +\infty$ by convention. Hence,

$$\begin{aligned}(\rho_\varepsilon^{(n)} * f)(x) &= \int_a^b \rho_\varepsilon^{(n)}(x-y)f(y)dy \\ &= \int_a^b \frac{d^n}{dy^n}\left((-1)^n \rho_\varepsilon(x-y)\right) f(y)dy \\ &= 0 \quad \forall x \in (a,b), \forall 0 < \varepsilon < \min\{(x-a)b^{-1}, (b-x)a^{-1}\}\end{aligned}$$

by assumption on f.

Fixing $I \subset (a,b)$ a compact interval which is not a singleton, we deduce $(\rho_\varepsilon^{(n)} * f)(x) = 0$ for all $x \in I$ and $\varepsilon > 0$ small enough (this depends only on I, a and b). Hence,

$$(\rho_\varepsilon * f)(x) = \sum_{k=0}^{n-1} c_{k,\varepsilon} x^k$$

almost everywhere on I.

To conclude the proof, it suffices to check that the coefficients $c_{k,\varepsilon}$ converge as $\varepsilon \to 0$. Pointwise convergence of $\rho_\varepsilon * f$ to f then implies that f is given by a polynomial $p_I(x)$ on I and we can use again the observation from the beginning of the proof to conclude.

Fix $x_1, ..., x_n \in I$ distinct for which $\lim_{\varepsilon \to 0} \rho_\varepsilon * f(x_i) = \lim_{\varepsilon \to 0} \sum_{k=0}^{n-1} c_{k,\varepsilon} x_i^k = f(x_i)$. This limit can be rewritten as

$$\lim_{\varepsilon \to 0} \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix} \cdot \begin{pmatrix} c_{0,\varepsilon} \\ c_{1,\varepsilon} \\ \vdots \\ c_{n-1,\varepsilon} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

The matrix on the left-hand side is an invertible Vandermonde matrix A which does not depend on ε. Applying A^{-1}, the vector of coefficients must converge. ☐

Corollary 1. *Fix $n \in \mathbb{N}_{\geq 0}$. The subspace $M = \{\phi^{(n)}(x) : \phi \in C_c^\infty(\mathbb{R})\} \subset L^2(\mathbb{R})$ is dense.*

Proof. The Du Bois-Reymond lemma implies that $M^\perp \subset L^2(\mathbb{R})$ contains only polynomials of degree $\leq n-1$. The only square-integrable polynomial is the trivial one, i.e. $M^\perp = \{0\}$, so M is dense. ☐

Remark 1. *If $-\infty < a < b < +\infty$, the subspace $M = \{\phi'(x) : \phi \in C_c^\infty((a,b))\} \subset L^2((a,b))$ is **not** dense. Indeed,*

$$\int_a^b \phi'(x)dx = \phi(b) - \phi(a) = 0 \quad \forall \phi \in C_c^\infty((a,b))$$

i.e. every element of M has zero mean. The embedding $L^2((a,b)) \subset L^1((a,b))$ implies that every element of $\overline{M}$ must have zero mean as well, so density cannot hold. In fact, the Du-Bois Reymond Lemma shows that $M^\perp$ is the set of constant functions.