Base change morphism:

Suppose we are given a Cartesian diagram of finite type k-schemes

$$\begin{tikzcd} X' \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

and a quasicoherent sheaf $\mathcal{F}$ on X.

We construct the base change morphism

$$\Phi^i_{Y'} : \psi^* R^i\pi_* \mathcal{F} \longrightarrow R^i\pi'_* \psi'^* \mathcal{F}$$

for the higher direct images.

$$\psi^* R^i\pi_* \mathcal{F} \rightarrow \psi^* R^i\pi_* (\psi'_* \psi'^* \mathcal{F})$$ (ψ'^*, ψ'_*) adjunction

$$= \psi^* R^i (\pi \circ \psi')_* (\psi'^* \mathcal{F})$$ (by definition)

$$= \psi^* R^i (\psi \circ \pi')_* (\psi'^* \mathcal{F})$$

$$\overset{\circledast}{\rightarrow} R^i \pi'_* (\psi'^* \mathcal{F})$$ ψ^* right exact & (ψ^*, ψ_*) adjunction.

$\circledast$ using the explicit definition

We define the morphism on each open $V \subset Y'$

$$H^0(V, \psi^* R^i (\psi \circ \pi')_* (\psi'^* \mathcal{F})) \rightarrow H^i(V, \psi^* \psi_* \pi'_* \psi'^* \mathcal{F})$$ ψ^* right exact.

$$\rightarrow H^i(V, \pi'_* \psi'^* \mathcal{F})$$ (ψ^*, ψ_*) adjunction

$$= H^0(V, R^i\pi'_* \psi'^* \mathcal{F})$$

⊛ using injective resolutions

Let $\psi'^* \mathcal{F} \to I^\bullet$ be an injective resolution.

$$\psi^* R^i(\psi \circ \pi')_* (\psi'^* \mathcal{F}) = \psi^* H^i(\psi_* \pi'_* I^\bullet)$$

$$\to H^i(\psi^* \psi_* \pi'_* I^\bullet) \qquad \psi^* \text{ right exact.}$$

$$\to H^i(\pi'_* I^\bullet) \qquad (\psi^*, \psi_*) \text{ adjunction.}$$

$$= R^i \pi'_* \psi'^* \mathcal{F}$$

Remark: In contrast to the incorrect treatment in the lecture we only used adjunctions for the horizontal maps ψ, ψ'.