MATH524 – Spring 2025

Problem Set: Week 8

Recall from last week's exercises that we can define the total variation distance as

$$TV(P,Q) = \frac{1}{2}\int |p-q|\,\mathrm{d}\nu = 1 - \int \min(p,q)\,\mathrm{d}\nu.$$

1. (**Pinsker's inequality**)

   (a) Show the following upper bound on the total variation distance:

   $$TV(P,Q) \leq \int (\sqrt{p}-\sqrt{q})^2\,\mathrm{d}\nu := h^2(P,Q)$$

   The function h^2 is also known as the Hellinger distance.

   (b) Using the result of part (a) as an intermediate step, show the following inequality:

   $$TV(P,Q)^2 \leq \mathrm{KL}(P,Q).$$

   Hint: *You may wish to use the fact that* $\log(1+x) \leq x$ *for* $x > -1$.

2. (**Le Cam's inequality**) Show the following relationship:

   $$\int \min(p,q)\,\mathrm{d}\nu \geq \frac{1}{2}\left(\int \sqrt{pq}\right)^2$$

3. (**Affinity, TV and KL**)

   (a) Using Jensen's inequality and Le Cam's inequality from Question 2, show that

   $$\int \min(p,q)\,\mathrm{d}\nu \geq \frac{1}{2}\exp(-\mathrm{KL}(P,Q))$$

   (b) Finally, conclude the following relationship between TV and KL:

   $$TV(P,Q) \leq 1 - \frac{1}{2}\exp(-\mathrm{KL}(P,Q)).$$