MATH524 – Spring 2025

Problem Set: Week 8

Recall from last week's exercises that we can define the total variation distance as

$$TV(P,Q) = \frac{1}{2}\int |p-q|\,\mathrm{d}\nu = 1 - \int \min(p,q)\,\mathrm{d}\nu.$$

1. (**Pinsker's inequality**)

   (a) Show the following upper bound on the total variation distance:

   $$TV(P,Q) \leq \int (\sqrt{p}-\sqrt{q})^2\,\mathrm{d}\nu := h^2(P,Q)$$

   The function h^2 is also known as the Hellinger distance.

   Solution: Let us define the following function:

   $$h^2(P,Q) = \int (\sqrt{p}-\sqrt{q})^2\,\mathrm{d}\nu.$$

   This is also known as the Hellinger distance. Using the absolute-value definition of TV and Cauchy-Schwarz,

   $$\begin{aligned} 2TV(P,Q) &= \int |p-q|\,\mathrm{d}\nu = \int |\sqrt{p}-\sqrt{q}||\sqrt{p}+\sqrt{q}|\,\mathrm{d}\nu \leq h(P,Q)\sqrt{\int |\sqrt{p}+\sqrt{q}|^2\,\mathrm{d}\nu} \\ &= h(P,Q)\sqrt{2+\int \sqrt{pq}\,\mathrm{d}\nu} = h(P,Q)\sqrt{4-h^2(P,Q)} \leq 2h(P,Q). \end{aligned}$$

   (b) Using the result of part (a) as an intermediate step, show the following inequality:

   $$TV(P,Q)^2 \leq \mathrm{KL}(P,Q).$$

   Hint: *You may wish to use the fact that* $\log(1+x) \leq x$ *for* $x > -1$.

   Solution: We will use the hint and the concavity of the log function.

   $$\begin{aligned} \mathrm{KL}(P,Q) &= \int p\log\frac{p}{q}\,\mathrm{d}\nu = 2\int p\log\frac{\sqrt{p}}{\sqrt{q}}\,\mathrm{d}\nu \\ &= -2\int p\log\Big(1+\frac{\sqrt{q}}{\sqrt{p}}-1\Big)\,\mathrm{d}\nu \geq -2\int p\Big(\frac{\sqrt{q}}{\sqrt{p}}-1\Big)\,\mathrm{d}\nu = h^2(P,Q). \end{aligned}$$

   The result now follows by squaring both sides of part (a).

2. (**Le Cam's inequality**) Show the following relationship:

   $$\int \min(p,q)\,\mathrm{d}\nu \geq \frac{1}{2}\left(\int \sqrt{pq}\right)^2$$

   Solution: We start by noticing the fact that

   $$\int \min(p,q)\,\mathrm{d}\nu + \int \max(p,q)\,\mathrm{d}\nu = 2.$$

Then,

$$\begin{aligned}
\left(\int \sqrt{pq}\right)^2 &= \left(\int \sqrt{\min(p,q)\max(p,q)}\right)^2 \\
&\le \int \min(p,q) \int \max(p,q) \\
&= \int \min(p,q)\left[2 - \int \min(p,q)\right] \\
&= 2\int \min(p,q) - \left(\int \min(p,q)\right)^2 \\
&\le 2\int \min(p,q),
\end{aligned}$$

where we use the fact that $(\int \min(p,q))^2 \ge 0$.

3. (**Affinity, TV and KL**)

   (a) Using Jensen's inequality and Le Cam's inequality from Question 2, show that

   $$\int \min(p,q)\,\mathrm{d}\nu \ge \frac{1}{2}\exp(-\mathrm{KL}(P,Q))$$

   Solution: Trivially, we only need to consider the case where $\mathrm{KL}(P,Q) < \infty$. By Jensen's inequality, we have

   $$\begin{aligned}
   \left(\int \sqrt{pq}\right)^2 &= \exp(2\log \int_{pq>0} \sqrt{pq}) \\
   &= \exp(2\log \int_{pq>0} p\sqrt{q/p}) \\
   &\ge \exp(2\int_{pq>0} p\log\sqrt{q/p}) \\
   &= \exp(-\int_{pq>0} p\log(\sqrt{p/q})^2) \\
   &= \exp(-\mathrm{KL}(P,Q)).
   \end{aligned}$$

   The conclusion follows from plugging in the expression into the bound from Le Cam's inequality.

   (b) Finally, conclude the following relationship between TV and KL:

   $$TV(P,Q) \le 1 - \frac{1}{2}\exp(-\mathrm{KL}(P,Q)).$$

   Solution: The result follows directly from part (a) and Scheffé's Theorem that was proven in the previous week's exercise.