MATH524 – Spring 2025

Problem Set: Week 7

1. (**Mean-absolute deviation forms GC class**) Consider the M-estimation example from Chapter 5 with the meand absolute deviation estimator:

$$m_\theta(x) = |x - \theta|$$

Assume further that $\mathbb{P}(x^2) < \infty$, $\Theta = [-1, 1]$, $d(x, y) = |x - y|$ and define $\mathcal{F} = \{m_\theta : \theta \in [-1, 1]\}$. In this exercise we will show that $\mathcal{F}$ forms a GC class.

   (a) Show that $m_\theta(x)$ is Lipschitz and identify the lipschitz constant.

   (b) By showing that the L_1 bracketing number can be bounded as

$$N_{[]}(\varepsilon, \mathcal{F}, L_1) < 2\frac{\text{diam}(\Theta)}{\varepsilon},$$

   show that $\mathcal{F}$ forms a GC class.
   Hint: *It may be helpful to identify a relationship between the bracketing and covering numbers for* Θ.

2. (**KL for Gaussians**) If $P = N(\theta, \sigma^2)$ and $Q = N(\mu, \sigma^2)$, show that

$$\text{KL}(P, Q) = \frac{(\theta - \mu)^2}{2\sigma^2}.$$

3. (**Scheffe's Theorem**) Define the total variation distance function as

$$TV(P, Q) = \sup_{A \in \mathcal{A}} |\int_A (p - q)\, \mathrm{d}\nu|$$

Show the following equality between total variation and affinity

$$TV(P, Q) = \frac{1}{2} \int |p - q|\, \mathrm{d}\nu = 1 - \int \min(p, q)\, \mathrm{d}\nu.$$

Hint: *Consider bounding the integral on sets of type* $\{x \in \mathcal{X} : q(x) \geq p(x)\}$