MATH524 – Spring 2025

Problem Set: Week 6

1. (**Subgaussian Chaining and Dudley's Entropy Integral**)

   Let $(\mathcal{T}, d)$ be a non-empty metric space and X_t be real-valued random variables for $t \in \mathcal{T}$. Suppose $\mathbb{E}[X_t] = 0$ for all $t \in \mathcal{T}$ and assume the subgaussian increments condition

   $$\mathbb{E}\big[\exp\big(\lambda(X_t - X_s)\big)\big] \le \exp\left(\frac{\lambda^2 d(s,t)^2}{2}\right)$$

   for all $s, t \in \mathcal{T}$. Assume that there is a countable subset $\mathcal{T}' \subseteq \mathcal{T}$ with $\sup_{t \in \mathcal{T}} X_t = \sup_{t' \in \mathcal{T}'} X_{t'}$ almost surely.

   (a) Show that if $t_1, \dots, t_N \in \mathcal{T}$ and $s_1, \dots, s_N \in \mathcal{T}$ satisfy $\max_{1 \le j \le N} d(s_j, t_j) \le \delta$ then

   $$\mathbb{E}\left[\max_{1 \le j \le N} \big(X_{t_j} - X_{s_j}\big)\right] \le \delta\sqrt{2\log N}.$$

   Hint: *you may wish to show first that if* $\lambda > 0$ *then* $\max_{1 \le j \le N} x_j \le \frac{1}{\lambda}\log\sum_{j=1}^N \exp(\lambda x_j)$ *for any* $x_j \in \mathbb{R}$.

   (b) For $\varepsilon > 0$ we say a subset $\mathcal{T}_\varepsilon \subseteq \mathcal{T}$ is an ε-cover of $(\mathcal{T}, d)$ if $\sup_{t \in \mathcal{T}} \inf_{s \in \mathcal{T}_\varepsilon} d(s,t) \le \varepsilon$. The covering number $N(\varepsilon, \mathcal{T}, d)$ is the cardinality of the smallest ε-cover. Show that if $\varepsilon \le \varepsilon'$ then $N(\varepsilon, \mathcal{T}, d) \ge N(\varepsilon', \mathcal{T}, d)$.

   (c) Define the diameter $D = \sup_{s,t \in \mathcal{T}} d(s,t)$. Show that if $D = \infty$ then $N(\varepsilon, \mathcal{T}, d) = \infty$ for all $\varepsilon > 0$.
   Hint: *You may choose to prove this result by contradiction.*

   For the remainder of this question we will assume $D < \infty$.

   (d) For $k \ge 0$ let $\mathcal{T}_k$ be a $(2^{-k}D)$-cover of $(\mathcal{T}, d)$ with cardinality $N_k = N\big(2^{-k}D, \mathcal{T}, d\big)$ and for all $t \in \mathcal{T}$ let $\pi_k(t) \in \mathcal{T}_k$ satisfy $d(t, \pi_k(t)) \le 2^{-k}D$. Using the result from part (a), show that $d(\pi_{k+1}(t), \pi_k(t)) \le 3 \cdot 2^{-k-1}D$ and deduce that

   $$\mathbb{E}\left[\sup_{t \in \mathcal{T}} \big(X_{\pi_{k+1}(t)} - X_{\pi_k(t)}\big)\right] \le 3 \cdot 2^{-k} D\sqrt{\log N_{k+1}}.$$

   (e) Show that $N_0 = 1$ and by considering the chain $t_{\pi_0(t)} \to t_{\pi_1(t)} \to \cdots \to t$ show that

   $$\mathbb{E}\left[\sup_{t \in \mathcal{T}} X_t\right] \le 6\sum_{k=1}^{\infty} 2^{-k} D\sqrt{\log N_k}.$$

   Hint: *first assume that* $\mathcal{T}$ *is finite, then extend to the countable setting, and finally to a general set* $\mathcal{T}$.

   (f) By considering Riemann sums, prove Dudley's entropy integral:

   $$\mathbb{E}\left[\sup_{t \in \mathcal{T}} X_t\right] \le 12\int_0^{\infty} \sqrt{\log N(\varepsilon, \mathcal{T}, d)}\, \mathrm{d}\varepsilon.$$