Empirical Processes

MAA110 - EPFL

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Exercise 1 (Parametric class of functions)**.** *Consider the class of parametric functions*

$$\mathcal{H} = \{h_\theta : x \in [0,1] \mapsto 1 - e^{-\theta x} \in \mathbb{R},\ \theta \in [0,1]\} \ ,$$

endowed with the uniform norm $\|f - g\|_\infty = \sup_{x\in[0,1]} |f(x) - g(x)|$.

1. *Prove that* $(\mathcal{H}, \|\cdot\|_\infty)$ *is a metric space*
2. *Define the covering number w.r.t. the uniform norm by* $N(\varepsilon) := N(\varepsilon, \mathcal{H}, \|\cdot\|_\infty)$*, for any* $\varepsilon > 0$*. Prove that it can be bounded as follows:*

$$N(\varepsilon) \leq 2 + \frac{1}{2\varepsilon} \ .$$

where we recall that $\lfloor\cdot\rfloor$ *is the floor function defined by: for any* $x \in \mathbb{R}$, $\lfloor x \rfloor = \max\{n \in \mathbb{Z},\ n \leq x\}$*.*
Hint: construct an ε*-cove of* $\mathcal{H}$ *based on a well-chosen grid of* $[0,1]$

Exercise 2. *Compute the* VC*-dimension of the following sets* $\mathcal{C}$*.*

1. $\mathcal{C} = \{(-\infty, x_1] \times \ldots \times (-\infty, x_d],\ (x_1, \ldots, x_d) \in \mathbb{R}^d\}$
2. $\mathcal{C} = \{[x_1, y_1] \times \ldots \times [x_d, y_d],\ (x_1, \ldots, x_d), (y_1, \ldots, y_d) \in \mathbb{R}^d,\ x_1 < y_1, \ldots, x_d < y_d\}$ *class of rectangles in* $\mathbb{R}^d$*.*

Exercise 3. *Provide an upperbound on the* VC*-dimension of the set of closed balls in* $\mathbb{R}^d$

$$\mathcal{C} = \left\{ \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d,\ \sum_{i\leq d} |x_i - a_i|^2 \leq b \right\},\ a_1, \ldots, a_d, b \in \mathbb{R} \right\}$$

Exercise 4. *Let* $\mathcal{C}$ *be a class of sets of* $\mathbb{R}^d$ *with finite* VC*-dimension* $\mathcal{V}$*, and of shattering coefficient* $m_n(\mathcal{C})$*, for all* $n \in \mathbb{N}^*$*. Prove that*

1. $m_n(\mathcal{C}) \leq (n+1)^{\mathcal{V}}$*, for all* $n \in \mathbb{N}^*$
2. $m_n(\mathcal{C}) \leq (ne/\mathcal{V})$*, for all* $n \geq \mathcal{V}$

Exercise 5 (Affine transformations)**.** *Let* $T \subseteq \mathbb{R}^d$*, and fix* $c \in \mathbb{R}^*$ *and a vector* $b \in \mathbb{R}^d$*. Prove that the Rademacher complexity of the affine transformation* $\Phi : t \mapsto ct + b$ *is proportional to* $\mathcal{R}(T)$ *as follows*

$$\mathcal{R}(\Phi(T)) = |c|\ \mathcal{R}(T) \ . \tag{1}$$

(Prove it without using Proposition 3.4.)

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Exercise 6 (Convex hull)**.** *Let $T \subseteq \mathbb{R}^d$, and define the convex hull of T by*

$$T' = conv(T) = \left\{ \sum_{i=1}^{m} c_i t_i,\ m \in \mathbb{N}, t_i \in T,\ c_i \geq 0,\ \|c\|_1 = 1 \right\} .$$

Then prove that the Rademacher process

$$\mathcal{R}(T) = \mathcal{R}(T') .$$