

Empirical Processes

MAA110 - EPFL

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Exercise 1 (Parametric class of functions). *Consider the class of parametric functions*

$$\mathcal{H} = \{h_\theta : x \in [0, 1] \mapsto 1 - e^{-\theta x} \in \mathbb{R}, \theta \in [0, 1]\} ,$$

endowed with the uniform norm $\|f - g\|_\infty = \sup_{x \in [0, 1]} |f(x) - g(x)|$.

1. *Prove that $(\mathcal{H}, \|\cdot\|_\infty)$ is a metric space*
2. *Define the covering number w.r.t. the uniform norm by $N(\varepsilon) := N(\varepsilon, \mathcal{H}, \|\cdot\|_\infty)$, for any $\varepsilon > 0$. Prove that it can be bounded as follows:*

$$N(\varepsilon) \leq 2 + \frac{1}{2\varepsilon} .$$

where we recall that $\lfloor \cdot \rfloor$ is the floor function defined by: for any $x \in \mathbb{R}$, $\lfloor x \rfloor = \max\{n \in \mathbb{Z}, n \leq x\}$.

Hint: construct an ε -cove of \mathcal{H} based on a well-chosen grid of $[0, 1]$

Exercise 2. *Compute the VC-dimension of the following sets \mathcal{C} .*

1. $\mathcal{C} = \{(-\infty, x_1] \times \dots \times (-\infty, x_d], (x_1, \dots, x_d) \in \mathbb{R}^d\}$
2. $\mathcal{C} = \{[x_1, y_1] \times \dots \times [x_d, y_d], (x_1, \dots, x_d), (y_1, \dots, y_d) \in \mathbb{R}^d, x_1 < y_1, \dots, x_d < y_d\}$ *class of rectangles in \mathbb{R}^d .*

Exercise 3. *Provide an upperbound on the VC-dimension of the set of closed balls in \mathbb{R}^d*

$$\mathcal{C} = \left\{ \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d, \sum_{i \leq d} |x_i - a_i|^2 \leq b \right\}, a_1, \dots, a_d, b \in \mathbb{R} \right\}$$

Exercise 4. *Let \mathcal{C} be a class of sets of \mathbb{R}^d with finite VC-dimension \mathcal{V} , and of shattering coefficient $m_n(\mathcal{C})$, for all $n \in \mathbb{N}^*$. Prove that*

1. $m_n(\mathcal{C}) \leq (n + 1)^\mathcal{V}$, for all $n \in \mathbb{N}^*$
2. $m_n(\mathcal{C}) \leq (ne/\mathcal{V})$, for all $n \geq \mathcal{V}$

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