# Empirical Processes

MAA110 - EPFL

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**Sub-Gussian random variables.** Recall that a centered r.v. $X$ is sub-Gaussian with variance parameter $\nu \geq 0$, if

$$\psi_X(\lambda) := \log \mathbb{E}[e^{\lambda X}] \leq \frac{\lambda^2 \nu}{2} \ . \tag{1}$$

The following exercises will provide other equivalent characterizations.

**Exercise 1** (Sub-Gaussian random variables)**.** *Consider $X$ to be sub-Gaussian with variance parameter $\nu \geq 0$. Prove that the following assertions are equivalent and give the explicit value of the constants $C_1, C_2, C_3, C_4$, up to numerical multiplicative constants $K$ that change for each of the statements ($C_i \leq K C_j$).*

*(i) sub-Gaussian tails. For all $t \geq 0$,*

$$\mathbb{P}(|X| \geq t) \leq 2 \exp\left\{-\frac{t^2}{C_1 \nu}\right\}$$

*(ii) sub-Gaussian moments. For all $k \in \mathbb{N}^*$*

$$\mathbb{E}[|X|^k]^{1/k} \leq C_2 \sqrt{\nu k}$$

*(iii) Super-exponential moments.*

$$\mathbb{E}\left[\exp\left\{\frac{X^2}{C_3 \nu}\right\}\right] \leq e$$

**Exercise 2** (Rademacher r.v.)**.** *Let a Rademacher r.v. $\varepsilon$, i.e., such that $\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = 1/2$. Prove that $\varepsilon$ is sub-Gaussian, and give the exact variance parameter.*

**Exercise 3** (Rademacher average)**.** *Let $n \in \mathbb{N}^*$ be fixed. Let an i.i.d. sequence of Rademacher variables $\varepsilon_1, \ldots, \varepsilon_n$, independent of the i.i.d. sample $X_1, \ldots, X_n$ composed of centered square-integrable random variables. The Rademacher average is defined by*

$$R_n = \sum_{i=1}^{n} \varepsilon_i X_i \ , \tag{2}$$

*Prove that the following inequality holds true, for all $t \geq 0$, and give the explicit value of the constant $c > 0$.*

$$\mathbb{P}\left\{|R_n| \geq t\right\} \leq 2 e^{-t^2/2c^2} \ . \tag{3}$$

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**Orlicz norms.** Orlicz spaces characterize broader classes of random variables, for which sub-Gaussian and sub-Exponential are special cases.

A class of Orlicz functions is composed by functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, s.t. $\psi(0) = 0$ and $\psi \to \infty$ when $t \to \infty$.

For a r.v. $X$, we define its Orlicz norm by

$$\|X\|_\psi := \inf\{t > 0, \quad \mathbb{E}[\psi(|X|/t)] \leq 1\} \ . \tag{4}$$

**Exercise 4** (Orlicz norm)**.** *1. Prove that $\|\cdot\|_\psi$ is convex on the space of random variables.*

*Hint: show that the bivariate function $(t, x) \in [0, \infty)^2 \mapsto t\psi(x/t)$ is convex in both coordinates.*

*2. Prove that $\|\cdot\|_\psi$ is a norm.*

**Exercise 5** (Properties)**.** *1. Notice that choosing $\psi(t) = t^p$, with $p > 0$ recovers the classical $L_p$-norm*

*2. Define $\psi_p(t) = e^{|x|^p} - 1$, with $p \in \{1, 2\}$. Prove that*

$$\|X\|_{\psi_2}^2 = \|X^2\|_{\psi_1}$$

*and that for any r.v. $Y$ independent of $X$*

$$\|XY\|_{\psi_1} \leq \|X\|_{\psi_2}\|Y\|_{\psi_2} \ .$$

**Exercise 6.** *Le $X$ be a r.v. with finite $\psi_1$-norm.*

*1. Prove that for all $t \geq 0$,*

$$\mathbb{P}(|X| \geq t) \leq 2\exp(-t/\|X\|_{\psi_1}) \ . \tag{5}$$

*2. Suppose now that $X$ is centered, and define $Z = X/\|X\|_{\psi_1}$. Prove that for all $|\lambda| \leq 1/2$,*

$$\psi_Z(\lambda) \leq 4\lambda^2 \tag{6}$$