Empirical Processes

MAA110 - EPFL

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Exercise 1 (Basic operations with outer measure)**.** Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space and let $X : \Omega \to \overline{\mathbb{R}}$ be an arbitrary random map. Suppose both measurable minimal majorant X^* and maximal minorant X_* exist. Let $B \subset \Omega$ be arbitrary. Prove the following assertions.

1. $P^*(B) = E^*1\{B\}$
2. There exists a measurable set $B^* \supset B$, such that $P(B^*) = P^*(B)$. In that case, show that $1\{B^*\} = (1\{B\})^*$
3. $(1\{B\})^* + (1\{\Omega - B\})_* = 1$

Exercise 2 (Basic operations with outer measure - 2)**.** Consider notations from Exercise 1. Let $Y : \Omega \to \overline{\mathbb{R}}$ a second arbitrary random map. Suppose both measurable minimal majorant Y^* and maximal minorant Y_* exist. Prove the following assertions.

1. $X_* + Y^* \leq (X + Y)^* \leq X^* + Y^*$ if X is measurable, then it holds with equalities
2. $(X - Y)^* \geq X^* - Y^*$
3. $(1\{X > t\})^* = 1\{X^* > t\}$ for any $t \in \mathbb{R}$. And similar $(1\{X > t\})_* = 1\{X_* > t\}$
4. $(X \vee Y)^* = X^* \vee Y^*$
5. $(X \wedge Y)^* \leq X^* \wedge Y^*$

Exercise 3 (Chebychev inequality)**.** let X be a r.r.v. defined on $(\Omega, \mathcal{A}, P)$, and let $h : \mathbb{R} \to \mathbb{R}$ be a monotone function, with extension to $\overline{\mathbb{R}}$.

1. Prove that $h(X^*) \geq (h(X))^*$.
2. Suppose for this question that h is cag on $[-\infty, \infty)$. Prove that the previous inequality is an equality.
3. Prove that, if X is nonnegative, and if $h : (0, \infty) \to (0, \infty)$, then for all $t > 0$

$$P^*(X \geq t) \leq \frac{E^*h(X)}{h(t)} \ .$$

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