Sharp phase transitions

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1 Intro

The last high-dimensional phenomenon that we will briefly look at is that of phase transitions. We will consider it in the setting of boolean functions on product measures on the hypercube $\{0,1\}^N$ or in other words, we will take i.i.d. $\{0,1\}$-valued random variables $X_1, X_2, \dots, X_N$ and consider random functions $f(X_1, \dots, X_n) \to \{0,1\}$.

Example 1.1. *As an example consider the so called majority function:* $M_n(x_1, \dots, x_n) = 1_{\sum_{i=1}^n x_i > n/2}$ *and the product measures where each* X_1 *is equal to* 1 *with probability* p*. We will measure the expected outcome:* $g_n(p) := \mathbb{E}_p(M_n)$ *as a function of* p*. For* $n = 1$*,* $g_n(p)$ *increases linearly from* 0 *to* 1*. But what happens at large* n*?*

We still see that $g_n(1/2) = \mathbb{E}_{1/2}(M_n) = 1/2$*. Further by concentration of measure* $\mathbb{P}_p(|\sum_{i=1}^n X_i - np| > t\sqrt{n}) \leq exp(-ct^2)$*. Thus, for every* $\epsilon > 0$*, we have that* $g_n(1/2+\epsilon) \to 1$ *and* $g_n(1/2 - \epsilon) \to 0$*. We can make this even more precise - for each* $\delta > 0$*, there is some* $t > 0$ *such that* $g_n(1/2 + t/\sqrt{n}) > 1 - \delta$ *and* $g_n(1/2 - t/\sqrt{n}) < \delta$ *for all* n *large enough. Verify that!*

On the other hand if we would consider the so called dictator function, given by say $D_n = 1_{x_1=1}$, the function $g_n(p) := \mathbb{E}_p(D_n)$ would of course not have any phase transition, but rather be linear for all n.

We will see that the general philosophy will be as follows: for symmetric increasing boolean functions the phase transitions are sharp in the following sense.

Definition 1.2. *A sequence of increasing boolean functions* (f_n) *undergoes a sharp threshold at* (p_n) *if there exists a sequence* (δ_n) *tending to* 0 *such that:*

$$g_n(p_n - \delta_n) \to 0 \quad \text{and} \quad g_n(p_n + \delta_n) \to 1.$$

By an increasing boolean function we mean here increasing w.r.t the natural partial order on $\{0,1\}^n$ - $\omega \leq \omega'$ if and only if $\omega_i \leq \omega'_i$ for all $i = 1 \dots n$.

As resources I recommend the book by van Handel and the notes by Duminil-Copin on "Sharp threshold phenomena in statistical physics".

2 Margulis-Russo lemma

Our aim is to understand the behaviour of $g_n(p) := \mathbb{E}_p(f_n)$ for a sequence of increasing boolean functions f_n. It comes out that one can calculate the derivative very nicely:

Lemma 2.1 (Margulis-Russo)**.** *We have that*

$$\frac{dg_n}{dp} = \sum_{i=1}^{n} \mathbb{E}_p(D_i(f_n)^2),$$

where as before $D_i(f_n) := \sup_z f_n(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) - \inf_z f_n(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$.

Remark 2.2. *In the case of boolean functions on the hypercube g one can see that* $\mathbb{E}_p(D_i(g)^2) = \mathbb{P}_p(g(\overline{X}) \neq g(\overline{X}^{(i)}))$, *where in $X^{(i)}$ we have flipped the $i-$th bit (verify this)!. This quantity* $\mathbb{P}_p(g(\overline{X}) \neq g(\overline{X}^{(i)}))$ *is also often called the $i-$th influence of g and denoted* $\mathrm{Inf}_i(g)$.

Proof. We write out

$$\frac{dg_n}{dp} = \frac{d}{dp} \sum_{\overline{x} \in \{0,1\}^n} p^{\sum x_i}(1-p)^{n-\sum x_i} f_n(\overline{x}).$$

Taking the derivative and noting that $x_i \in \{0, 1\}$ we can write this conveniently as

$$\sum_{\overline{x} \in \{0,1\}^n} \sum_{i=1}^{n} x_i p^{\sum_{j \neq i} x_j}(1-p)^{n-1-\sum_{i \neq j} x_j} f_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) -$$
$$- \sum_{\overline{x} \in \{0,1\}^n} \sum_{i=1}^{n} (1-x_i) p^{\sum_{j \neq i} x_j}(1-p)^{n-1-\sum_{i \neq j} x_j} f_n(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

which is just equal to

$$\sum_{i=1}^{n} \left(\mathbb{E}_p(f_n(X_1, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n) - \mathbb{E}_p(f_n(X_1, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n)) \right),$$

which in turn equals $\sum_{i=1}^{n} \mathbb{E}_p((D_i f_n)^2)$ as desired. □

Now recall that by Efron-Stein we have that

$$\mathrm{Var}_p(f_n) \leq \sum_{i=1}^{n} \mathbb{E}_p((D_i f_n)^2), \tag{2.1}$$

giving the inequality $\frac{dg_n}{dp} \geq \mathrm{Var}_p(f_n)$. Thus we already see that if f_n is not constant, then g_n is a strictly increasing function. Further, any improvement on (2.1) will make the derivative larger and thus the transitions sharper. Let us make this precise.

Lemma 2.3. *Suppose that $C_n Var_p(f_n) \leq \sum_{i=1}^n \mathbb{E}_p((D_i f_n)^2)$ and let $p_c = p_c(n)$ be such that $g_n(p_c) = 1/2$. Then for every $\delta > 0$ we have that $g_n(p_c + \delta) \geq 1 - \exp(-C_n\delta)$ and $g_n(p_c - \delta) < \exp(-C_n\delta)$.*

Proof. The key observation is the following: since f_n is $\{0,1\}$-valued, $\mathrm{Var}_p[f_n] = \mathbb{P}_p[f_n = 1] - \mathbb{P}_p[f_n = 1]^2 = g_n(p)(1 - g_n(p))$. Thus,

$$\frac{dg_n(p)}{dp} \geq C_n g_n(p)(1 - g_n(p))$$

can be rewritten as

$$(\log \frac{g_n(p)}{1 - g_n(p)})' \geq C_n.$$

But now we can integrate this inequality between p_c and $p_c+\delta$ to obtain $\log \frac{g_n(p_c+\delta)}{1-g_n(p_c+\delta)} \geq C_n\delta$. Taking the exponent this gives

$$g_n(p_c + \delta) \geq 1 - \frac{1}{1 + \exp(C\delta)} \geq 1 - \exp(-C_n\delta)$$

as desired. The other half works exactly in the same way. □

Thus we need to just understand in which circumstances and by how much we can hope to improve on the Poincaré inequality. Before doing that let us look at again at the example of the majority function and try to rederive the sharpness by improving on the Efron-Stein.

3 Majority function is the sharpest

Example 3.1. *We can calculate.*

On the one hand $Var_p(M_n) = \mathbb{P}(M_n > n/2)(1 - \mathbb{P}(M_n > n/2))$ is equal to $1/4$ at $p = 1/2$, and is smaller at other p.

On the other hand a simple calculation shows that the total influence is of order $\sqrt{n}$ (do it!).

And thus we find the idea that somehow a rapid change happens in the window of size $1/\sqrt{n}$.

In fact the majority function offers the sharpest transition for all increasing boolean functions g which are "fair" in the sense that $\mathbb{E}_{1/2}(g) = 1/2$.[1]

Lemma 3.2. *Let f_n be a a fair increasing boolean function which is fair in the sense that $\mathbb{E}_{1/2}(f_n) = 1/2$. Then $\mathbb{E}_p(f_n) \geq \mathbb{E}_p(M_n)$ for all $p < 1/2$ and $\mathbb{E}_p(f_n) \leq \mathbb{E}_p(M_n)$ for all $p > 1/2$.*

[1] To understand the notion of "fair" think in terms of voting schemes.

Proof. The property of fairness is equivalent to $\sum_{\overline{x}} f_n(\overline{x}) = \sum_{\overline{x}} M_n(\overline{x})$.

Further if $f_n \neq M_n$, as f_n is also fair there must exist both some $\overline{y}$ with $\sum_{i=1}^n y_i < n/2$ and $f_n(\overline{y}) = 1$ and some $\overline{z}$ with $\sum_{i=1}^n z_i > n/2$ and $f_n(\overline{z}) = 0$. We can further suppose that for all $\overline{y}' \leq \overline{y}$ we have $f_n(\overline{y}') = 0$ and for all $\overline{z}' \geq \overline{z}$ we have $f_n(\overline{z}') = 1$.

We will now see that swapping the values at $\overline{z}$ and $\overline{y}$ will still give us a fair function, that we thereby increase $\mathbb{E}_p(f)$ for $p > 1/2$, decrease it for $p < 1/2$ and further we get closer to the majority function.

Indeed, defining $\tilde{f}$ by letting $\tilde{f}(\overline{x}) = f(\overline{x})$ for $\overline{x} \notin \{\overline{y}, \overline{z}\}$ and setting $\tilde{f}(\overline{y}) = 0$ and $\tilde{f}(\overline{z}) = 0$, we see directly from the definition that $\mathbb{E}_p(\tilde{f}) \geq \mathbb{E}_p(f)$ for $p > 1/2$ and $\mathbb{E}_p(\tilde{f}) \leq E_p(f)$ for $p < 1/2$.

But now either $\tilde{f}$ is the majority function, or we can iterate the same procedure. As with every step there are less elements with $\sum_{i=1} x_i > n/2$ for which the function value is 0, this procedure has to end with the majority function and the lemma follows.

□

4 Sharp transition for symmetric functions

We now state a key result on bounding the variance (whose proof we discuss next time) and then deduce a nice generic threshold result from this.

Theorem 4.1 (Talagrand)**.** *There exists a constant $c > 0$ such that for any $p \in [0,1]$ and $n \in \mathbb{N}$, the following holds. For any increasing boolean function $f\colon \{0,1\}^n \to \{0,1\}$,*

$$\mathrm{Var}_p(f) \leq c \log\left(\frac{2}{p(1-p)}\right) \sum_{i=1}^n \frac{\mathrm{Inf}_i[f]}{\log\left(\frac{1}{\mathrm{Inf}_i[f]}\right)}.$$

Using this we can prove something rather beautiful.

Theorem 4.2 (Friedgut-Kalai)**.** *Suppose that an increasing boolean function f_n is invariant under a transitive family of permutations, i.e. $f_n(\overline{x}) = f_n(\sigma(\overline{x}))$ for any such permutation σ.*

Then for any $\delta > 0$ $\mathbb{E}_{p_c+\delta}(f_n) > 1 - n^{-\tilde{c}\delta}$ and $\mathbb{E}_{p_c-\delta}(f_n) < n^{-\tilde{c}\delta}$ for some constant $\tilde{c} > 0$ that does not depend on n.

Proof. We need to just obtain the inequality $\tilde{c} \log n \mathrm{Var}_p(f_n) \leq \sum_{i=1}^n \mathrm{Inf}_i[f_n]$.

Observe that because of the symmetry all influences are equal. We consider two different cases.

Either $\mathrm{Inf}_i[f_n] \geq \frac{\log n}{n}$ for all i, in which case

$$\sum_{i=1}^n \mathrm{Inf}_i[f_n] \geq \log n \geq \log n \mathrm{Var}_p(f_n).$$

Or $\mathrm{Inf}_i[f_n] < \frac{\log n}{n}$ for all i. In this case, $\log(1/\mathrm{Inf_i}[f_n]) \geq \log n - \log\log n$. Thus, from Talagrand's inequality

$$\mathrm{Var}_p(f_n) \leq c\left(\log\frac{1}{p(1-p)}\right)\frac{2}{\log n - \log\log n}\sum_{i=1}^{n}\mathrm{Inf}_i[f_n],$$

which gives some constant $\tilde{c}$ such that

$$\tilde{c}\log n \mathrm{Var}_p(f_n) \leq \sum_{i=1}^{n}\mathrm{Inf}_i(f_n).$$

This constant does depend on p, but is compact on any interval away from 0 and 1 and crucially is independent of n.

$\square$